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Bi-Elliptic Transfer with Plane Change

MAY 1965

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Prepared for COMMANDER SPACE SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
Los Angeles, California

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BI-ELLIPTIC TRANSFER WITH PLANE CHANGE

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This technical documentary report has been reviewed and is approved for publication and dissemination. The conclusions and findings contained herein do not necessarily represent an official Air Force position.

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ABSTRACT

By the calculations presented, the minimum total velocity increment required for bi-elliptic transfer between non-coplanar circular orbits is obtained. The maneuver considered is the following: A vehicle in circular orbit at altitude h_i (radius r_i) applies an impulsive velocity ΔV_1 at the line of nodes. The effect of the application of ΔV_1 causes a plane change of amount α_1 and a transfer ellipse to a given transfer altitude h_t (radius r_t) is established. When the vehicle reaches h_t, a second impulsive velocity change ΔV_2 simultaneously changes the plane by amount α_2 and initiates a transfer ellipse to the altitude h_f (radius r_f) of the target orbit. A last impulse ΔV_3 changes the plane by amount α_3 and circularizes the orbit at altitude h_f, placing the vehicle in the final (target) circular orbit.

Studies were made of the choice of plane change angles α_1, α_2, and α_3, which minimizes $\Delta V_T \equiv \Delta V_1 + \Delta V_2 + \Delta V_3$ for given values of h_i, h_t, h_f and total plane change angle $\theta = \alpha_1 + \alpha_2 + \alpha_3$. Several limiting relations were obtained for α_1, α_2, and α_3; they are dependent on either the ratio r_t/r_i alone, or the ratios r_t/r_i and r_t/r_f, and are independent of θ.

CONTENTS

FIGURES

I. INTRODUCTION

Problems dealing with orbital transfer are of considerable current importance. For non-coplanar orbits, it is particularly important to minimize (or nearly minimize) the velocity expenditure necessary to accomplish the specified plane change. In this report, the minimum velocity expenditure for bi-elliptic transfer between non-coplanar circular orbits will be obtained.

One use of bi-elliptic transfers is in the physical problem where rendezvous is desired between vehicles in non-coplanar circular orbits. The line of intersection of the two orbit planes will be referred to as the line of nodes. The altitudes above the surface of the earth of the inner and outer orbits are respectively denoted as h_i and h_f and the plane change angle is θ.

In this study, plane change angles in the range $0 < \theta \leq \pi/2$ will be considered. It should, however, be noted that the various conclusions relative to the division of plane change angle and the equations for obtaining plane change angle remain valid for $\pi/2 \leq \theta \leq \pi$. The interest in plane change angles greater than $\pi/2$ did not appear sufficient to warrant the necessary additional length and complexity in the various proofs. A study of bi-elliptic transfers with $\theta = 0$ can be found in Reference 1.

The bi-elliptic transfer is initiated by applying, at the line of nodes, a velocity increment ΔV_1, which transfers the rendezvous vehicle from the circular orbit at altitude h_i into an elliptical orbit with apsidal altitudes h_i and h_t, while simultaneously rotating the orbit plane through an angle α_1.

A second velocity impulse ΔV_2, applied at altitude h_t, simultaneously transfers the rendezvous vehicle into a second elliptical orbit with apsidal altitudes h_t and h_f, while rotating the plane through angle α_2. The rendezvous maneuve is completed by the application of velocity increment ΔV_3, which transfers

the rendezvous vehicle from the second elliptical orbit into the final circular orbit. Evidently the final plane change angle is $\alpha_3 = \theta - \alpha_1 - \alpha_2$. Figure 1 is a sketch of the bi-elliptic transfer maneuver when $r_i < r_t < r_f$.

For a rendezvous mission, the choice of h_t depends upon the relative phasing of the two orbital vehicles. References 2 and 3 discuss the selection of h_t for a given rendezvous mission and, in addition, present an alternative three-dimensional transfer scheme. Since h_t is determined by methods described in the references, the present analysis can be considered a treatment of a pure transfer problem in which h_t is given along with h_i, h_f, and θ.

Three separate cases must be considered, which can be classified as follows:*

$h_i < h_f \leq h_t$ (outer bi-elliptic transfer)

$h_i \leq h_t < h_f$ (intermediate bi-elliptic transfer)

$h_t < h_i < h_f$ (inner bi-elliptic transfer). (1)

The velocity required to transfer from h_f to h_t to h_i is equal to the velocity necessary to make the transfer in the reverse order. Therefore, the three cases in Eq. (1) cover all logically possible cases, with the possible relabeling of h_i and h_f.

It is desired to determine what portion of the total plane change θ should be accomplished at each of the altitudes concerned (h_i, h_t, and h_f) in order to minimize the total velocity expenditure. The plane change angles at h_i, h_t, and h_f are respectively denoted by α_1, α_2, and α_3. The corresponding

*The degenerate case $h_i = h_f$ is not treated here.

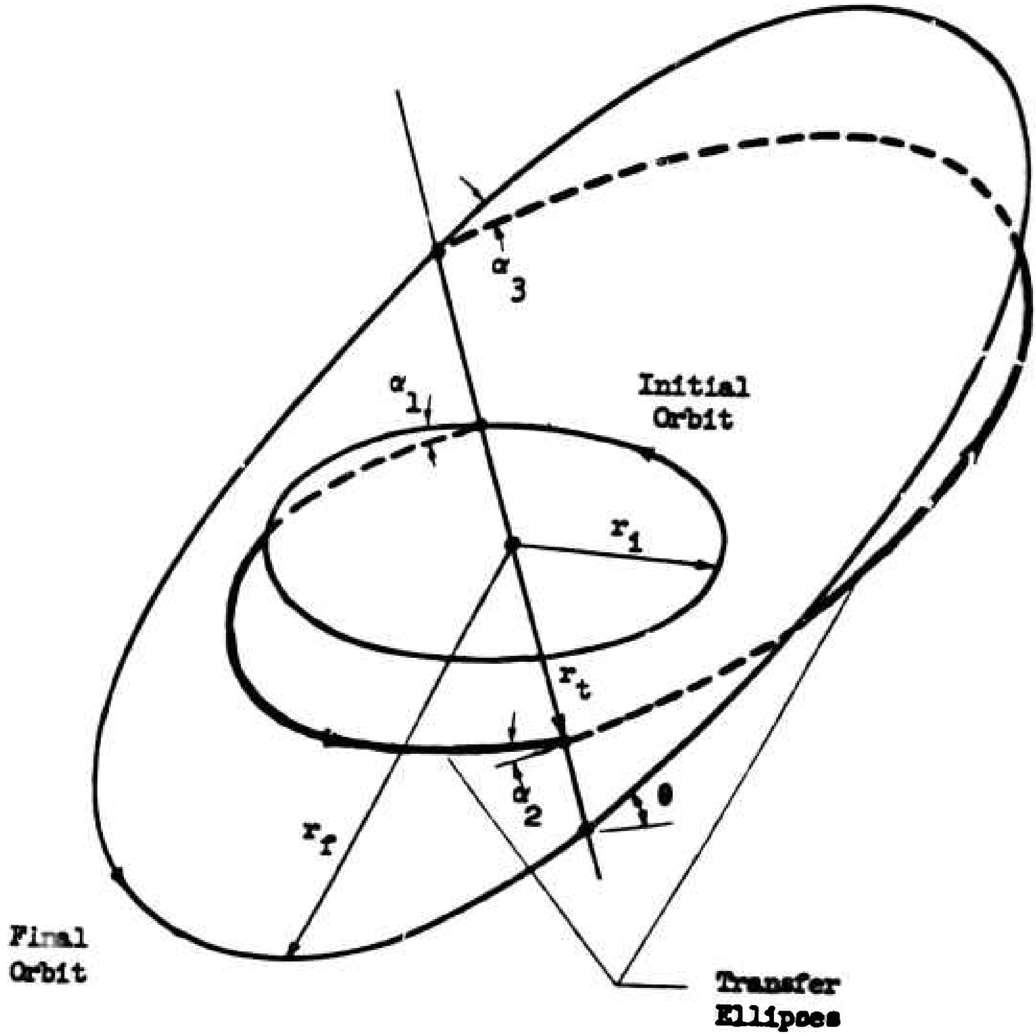

Figure 1. The General Bi-elliptic Transfer Maneuver

geocentric radii are defined as

$$r_i = r_o + h_i \quad , \quad r_t = r_o + h_t \quad , \quad r_f = r_o + h_f \tag{2}$$

where r_o is the radius of the attracting sphere.*

Section II describes the mathematical technique to be utilized in the subsequent analysis. Pertinent results are presented and briefly discussed; additional graphical results are presented in Section VII.

*For present purposes, the attracting sphere will be assumed to be the earth.

II. SUMMARY OF RESULTS

Bi-elliptic transfers with plane changes whose total velocity requirements are minimized all possess two (of the three) angles that are bounded. For the outer bi-elliptic transfer, they are the first and third plane change angles (α_1 and α_3). For the inner bi-elliptic transfer, they are the first and second angles (α_1 and α_2). Either α_1 and α_2, or α_1 and α_3, are bounded for the intermediate bi-elliptic transfer.

A numerical upper bound can, in fact, be placed on some of the above angles as follows:

a. The angle α_1 is less than an angle K, which is approximately 5. 30 deg for either the outer or the intermediate bi-elliptic transfer.

b. For an outer bi-elliptic transfer $\alpha_3 < K \approx 5.3$ deg.

c. For an inner bi-elliptic transfer $\alpha_2 < K \approx 5.3$ deg.

The other angles cited are bounded by functions of the orbital radii and the transfer radius.

The Hohmann transfer is investigated as a limiting case of a bi-elliptic transfer.

If α_1, α_2, and α_3 are respectively the plane change angles at r_i, r_t, and r_f, then the velocity increment ΔV_T, necessary for the bi-elliptic transfer, can be expressed as

$$\frac{\Delta V_T}{V_{ci}} = f(x, y, \alpha_1, \alpha_2, \alpha_3)$$

where V_{ci} is the circular orbit velocity at radius r_i and

$$x = \frac{r_t}{r_i} \quad , \quad y = \frac{r_t}{r_f} \qquad (r_f > r_i) \quad .$$

The total plane change θ is simply the sum $\alpha_1 + \alpha_2 + \alpha_3$.

The values of α_1, α_2, and α_3, which minimize $\Delta V_T / V_{ci}$, are respectively denoted as α_{1s}, α_{2s}, and α_{3s}.* If $r_t \geq r_f$, the angles α_{1s} and α_{3s} are obtained as the unique solutions of the equation

$$d\left(\frac{\Delta V_T}{V_{ci}}\right) = \frac{\partial}{\partial \alpha_1}\left(\frac{\Delta V_T}{V_{ci}}\right) d\alpha_1 + \frac{\partial}{\partial \alpha_3}\left(\frac{\Delta V_T}{V_{ci}}\right) d\alpha_3 = 0 \quad . \tag{3}$$

It is furthermore found that for all θ, $\alpha_{1s} \leq \bar{\alpha}_1$ and $\alpha_{3s} \leq \bar{\alpha}_3$ where

$$\bar{\alpha}_1 = \text{arc cos}\left[\sqrt{\frac{2}{x^3(1+x)}} + \frac{(x-1)}{x}\sqrt{\frac{x+2}{x}}\right] ,$$

$$\bar{\alpha}_3 = \text{arc cos}\left[\sqrt{\frac{2(1+y)}{y^3(1+x)^2}} + \sqrt{\frac{(1+x)^2 y^3 + 2(1+y) - (1+x)(1+3y)y}{y^3(1+x)^2}}\right]^{**} . \tag{4}$$

The following properties of α_{1s} and α_{3s} can be deduced from Eq. (4), when $r_t \geq r_f$:

$$\alpha_{1s} \leq \bar{\alpha}_1 < K \doteq 5.30°$$

$$\alpha_{3s} \leq \bar{\alpha}_3 < K$$

$$\bar{\alpha}_3 < \bar{\alpha}_1 \text{ for all } y < x$$

$$\bar{\alpha}_3 = \bar{\alpha}_1 \text{ for } y = x \qquad (r_i = r_f)$$

---

*Since $\alpha_{1s} + \alpha_{2s} + \alpha_{3s} = \theta$, any two of the above three angles determine the third.

**All angles are between 0° and 180° or 0 and π.

$$\alpha_{1s} = \bar{\alpha}_1 = 0, \qquad \text{when } x = 1$$

$$\alpha_{3s} = \bar{\alpha}_3 = 0, \qquad \text{when } y = 1$$

when

$$x = 1 \quad , \quad x \geq y = 1$$

and, therefore,

$$\alpha_{3s} = 0, \qquad \text{when } x = 1 \quad .$$

The properties when $x = y = 1$ can evidently be deduced on purely physical grounds. The case $x > 1$, $y = 1$ is seen to be the Hohmann transfer with plane change. The above result ($\alpha_{3s} = 0$ when $y = 1$) implies that no ΔV_T savings can be achieved by making part of the total plane change after circularizing at the final orbit altitude.

The subsequent study of the case $x = 1$, $y < 1$ indicates that no plane change should be initiated prior to the entrance into the transfer ellipse. Therefore, for a Hohmann transfer the total plane change maneuver should be divided between the initial impulse removing the vehicle from the inner orbit and the final impulse placing the vehicle in the outer orbit. Furthermore, the initial plane change angle, resulting in the minimum value of ΔV_T for a Hohmann transfer, is less than $\bar{\alpha}_1$.

When $r_t \leq r_i$, it is found advantageous to solve Eq. (3) for α_{1s} and α_{2s}, rather than α_{1s} and α_{3s}. If $r_t \leq r_i$, then $\alpha_{1s} \leq \alpha_1'$ and $\alpha_{2s} \leq \alpha_2'$ where

$$\alpha_1' = \text{arc cos}\left[\sqrt{\frac{2(1+x)y^4}{x^3(1+y)^2}} + \sqrt{\frac{x^3(1-y)(1+y-xy^2-y^2)}{y^3}}\right] \tag{5}$$

(cont.)

$$\alpha_2' = \text{arc cos}\left[\sqrt{\frac{(1+x)y^4}{x^4(1+y)}} + \sqrt{(1-y)(1+y-xy^2-y^2)}\right] \quad . \qquad (5)$$

The following properties of α_{1s} and α_{2s} can be deduced from Eq. (5), when $r_t \leq r_i$:

$$\alpha_{1s} \leq \alpha_1' \leq \alpha_{1m} = \text{arc cos}\sqrt{\frac{2x}{1+x}}$$

$$\alpha_{1s} = \alpha_1' = 0, \qquad \text{when } \frac{x}{y} = \frac{r_f}{r_i} = \infty$$

$$\alpha_1' = \alpha_{1m}, \qquad \text{when } y = x(r_i = r_f)$$

$$\alpha_{1s} = \alpha_1' = 0, \qquad \text{when } x = 1(r_t = r_i)$$

$$\alpha_{2s} \leq \alpha_2' < K \simeq 5.30°$$

$$\alpha_{2s} = \alpha_2' = 0, \qquad \text{when } x = \frac{r_t}{r_i} = 0$$

$$\alpha_{2s} = \alpha_2' = 0, \qquad \text{when } y = x(r_i = r_f)$$

$$\alpha_2' \text{ is maximized}, \qquad \text{when } x = 1(r_t = r_i) \quad .$$

It is interesting to note that the angles $\bar{\alpha}_1$, $\bar{\alpha}_3$, and α_2' are all bounded by the same angle $K = 5.30°$.

For $r_i < r_t < r_f$, $\alpha_{1s} \leq \bar{\alpha}_1$ (see Eq. (4). Furthermore, when $r_i < r_t < r_f$

$$\alpha_{2s} < \alpha_{2m} = \text{arc cos}\sqrt{\frac{1+y}{1+x}}$$

if

$$x > \frac{1 + y}{y^2} - 1 \quad .$$

Similarly,

$$\alpha_{3s} < \alpha_{3m} = \text{arc cos} \sqrt{\frac{2y}{1 + y}}$$

if

$$x \leq \frac{1 + y}{y^2} - 1 \quad .$$

The present analysis places bounds on the independent variables regardless of the values of r_i, r_t, and r_f. The two variables chosen (from the three variables α_{1s}, α_{2s}, and α_{3s}) are in turn determined by the values of x and y.

III. TRANSFER ALTITUDE h_t GREATER THAN FINAL ALTITUDE h_f

Consideration will begin with the first classification in Eq. (1), with which, in summary, the following maneuver is associated. A transfer is initially made from a circular orbit of altitude h_i to an elliptic orbit with apogee altitude h_t. The angle between the two orbit planes is α_1.

It can be shown that the velocity increment ΔV_1, necessary to accomplish this initial maneuver, is given by

$$\frac{\Delta V_1}{V_{ci}} = \sqrt{1 - 2H_1 \cos\ \alpha_1 + H_1^2} \tag{6}$$

The functions H_1 and V_{ci} are defined as

$$V_{ci} = \sqrt{\frac{\mu}{r_i}} \quad , \qquad H_1 = \sqrt{\frac{2x}{1+x}} \tag{7}$$

where $x = r_t/r_i$, μ is the force constant for the earth, and the function V_{ci} is the circular orbit velocity at altitude h_i.

The second transfer maneuver is initiated at altitude h_t. It consists of transferring from an orbit with apogee at h_t and perigee at h_i to one with apogee at h_t and perigee at h_f, accompanied by a plane change α_2 at h_t.

The velocity ΔV_2 corresponding to the second maneuver is given by

$$\frac{\Delta V_2}{V_{ci}} = H_2\sqrt{1 - 2H_3 \cos\ \alpha_2 + H_3^2} \tag{8}$$

The functions H_2 and H_3 are defined as

$$H_2 = \sqrt{\frac{2}{x(1+y)}} \quad , \quad H_3 = \sqrt{\frac{1+y}{1+x}} \tag{9}$$

where $x = r_t/r_i$ and $y = r_t/r_f$.

The final maneuver consists of transferring from an elliptical orbit with apogee at h_t and perigee at h_f to a circular orbit at altitude h_f, accompanied by a plane change α_3. The velocity expenditure ΔV_3 can be given as

$$\frac{\Delta V_3}{V_{ci}} = H_4 \sqrt{1 - 2H_5 \cos \alpha_3 + H_5^2} \tag{10}$$

where

$$H_4 = \sqrt{\frac{y}{x}} \quad , \quad H_5 = \sqrt{\frac{2y}{1+y}} \quad \text{and } \alpha_3 = \theta - \alpha_1 - \alpha_2 \tag{11}$$

It follows from the above analysis that the velocity increment ΔV_T for the bi-elliptic maneuver is given by

$$\frac{\Delta V_T}{V_{ci}} = \frac{\Delta V_1}{V_{ci}} + \frac{\Delta V_2}{V_{ci}} + \frac{\Delta V_3}{V_{ci}} \tag{12}$$

where the three components are given in Eqs. (6) through (11). It is desired to obtain the minimum ΔV_T for given values of x, y, and θ. Equations (6) through (12) can be shown to apply for arbitrary values of h_i, h_f, and $h_t (h_i \le h_f)$.

It will be shown that the minimum value of ΔV_T can be obtained from an examination of the partial derivatives of ΔV_T with respect to α_1 and α_3. A method will be developed to obtain this minimum.

From Eqs. (6), (8), (10), and (12), the partial derivatives of ΔV_T, with respect to α_1 and α_3, are given by

$$\frac{1}{V_{ci}} \frac{\partial \Delta V_T}{\partial \alpha_1} = \frac{\partial}{\partial \alpha_1}\left(\frac{\Delta V_1}{V_{ci}}\right) + \frac{\partial}{\partial \alpha_1}\left(\frac{\Delta V_2}{V_{ci}}\right)$$

$$= \frac{H_1 \sin \alpha_1}{\sqrt{1 - 2H_1 \cos \alpha_1 + H_1^2}} - \frac{H_2 H_3 \sin \alpha_2}{\sqrt{1 - 2H_3 \cos \alpha_2 + H_3^2}}$$

$$\frac{1}{V_{ci}} \frac{\partial \Delta V_T}{\partial \alpha_3} = \frac{H_4 H_5 \sin \alpha_3}{\sqrt{1 - 2H_5 \cos \alpha_3 + H_5^2}} - \frac{H_2 H_3 \sin \alpha_2}{\sqrt{1 - 2H_3 \cos \alpha_2 + H_3^2}} \tag{13}$$

where $\alpha_2 = \theta - \alpha_1 - \alpha_3$.

Examination of Eq. (13) shows that (when $y \neq x$ and $y \neq 1$):

$$\frac{\partial \Delta V_T}{\partial \alpha_1} < 0 \text{ at } \alpha_1 = 0$$

$$\frac{\partial \Delta V_T}{\partial \alpha_1} > 0 \text{ at } \alpha_1 = \theta - \alpha_3 \quad ;$$

$$\frac{\partial \Delta V_T}{\partial \alpha_3} < 0 \text{ at } \alpha_3 = 0 \quad ,$$

$$\frac{\partial \Delta V_T}{\partial \alpha_3} > 0 \text{ at } \alpha_3 = \theta - \alpha_1 \quad . \tag{14}$$

The first pair of values in Eq. (14) implies that, for any $\alpha_3 (0 \le \alpha_3 \le \theta)$, there exists an α_1, which minimizes ΔV_T for the chosen value of α_3. It further follows that

$$f(\alpha_1, \alpha_3) = \frac{\partial \Delta V_T}{\partial \alpha_1} = 0 \tag{15}$$

at the above minimum; i.e., the solution of Eq. (15) yields the value of α_1 which minimizes ΔV_T for any chosen value of α_3. Similarly, the second pair of values in Eq. (14) implies that there exists an α_3, which minimizes ΔV_T for any given $\alpha_1 (0 \le \alpha_1 \le \theta)$. The locus of these α_3 values is obtained by finding the roots of $g(\alpha_1, \alpha_3) = \partial \Delta V_T / \partial \alpha_3$, i.e., by solving the equation

$$g(\alpha_1, \alpha_3) = \frac{\partial \Delta V_T}{\partial \alpha_3} = 0 \tag{16}$$

for any given value of α_1. The minimum value of ΔV_T is obtained by solving the following system of equations for α_1 and α_3 (see Eqs. (15) and (16)):

$$f(\alpha_1, \alpha_3) = 0$$

$$g(\alpha_1, \alpha_3) = 0 \tag{17}$$

if a solution exists.

Physical reasoning indicates that, for $h_t > h_f$, the values of α_1 and α_3 which minimize ΔV_T, should be small. This fact alone is usually sufficient for obtaining iterative solutions to Eq. (17) (see Reference 4). The investigation that follows provides further insight into the problem of minimizing ΔV_T for all values of h_i, h_t, and $h_f (h_i \neq h_f)$.

The existence, uniqueness, and singularities of solutions of Eq. (17) will be discussed in Section IV. In order to accomplish this investigation effectively, attention is first directed to the function

$$F(\alpha) = \frac{H \sin \alpha}{\sqrt{1 + H^2 - 2H \cos \alpha}} \quad . \tag{18}$$

IV. PROPERTIES OF THE FUNCTION $F(\alpha)$

The following properties of $F(\alpha)$ are deduced from inspection:

$$F(\alpha) \geq 0 \text{ for } H \geq 0$$

$$F(0) = 0 \text{ for } H \neq 1$$

$$F\left(\frac{\pi}{2}\right) = \frac{H}{\sqrt{1 + H^2}} \tag{19}$$

Setting the first derivative of $F(\alpha)$ equal to zero shows that the maximum value of $F(\alpha)$ occurs when $\alpha = \alpha_m$ where

$$\cos \alpha_m = \frac{1 + H^2}{2H} \pm \sqrt{\left(\frac{1 + H^2}{2H}\right)^2 - 1}$$

$$= \frac{1 + H^2}{2H} \pm \frac{\sqrt{(1 - H^2)^2}}{2H} \tag{20}$$

Since $(1 + H^2)/2H \geq 1$ for all H, it follows that the maximum value of $F(\alpha)$ occurs at $\alpha = \alpha_m$ where

$$\cos \alpha_m = \frac{1}{H} \text{ for } H \geq 1$$

$$\cos \alpha_m = H \text{ for } H < 1 \tag{21}$$

Substitution of Eq. (21) into Eq. (18) yields the maximum value F_{max} of $F(\alpha)$ as

$$F_{max} = 1 \text{ for } H \geq 1$$

$$F_{max} = H \text{ for } H < 1 \tag{22}$$

V. SOLUTION OF THE EQUATIONS FOR THE MINIMUM TOTAL VELOCITY

This section will reformulate the system of equations in Eq. (17) (Eqs. (15) and (16)) in a form more suitable for iterative solution. The task of iteratively solving Eq. (17) will be further facilitated by obtaining the upper bounds on the angles $a_1 = a_{1s}$ and $a_3 = a_{3s}$, which minimize ΔV_T.

Substitution of the first equation of (13) into Eq. (15), when $\partial \Delta V_T / \partial a_1 = 0$, yields

$$G(a_1) \equiv \frac{H_1 \sin a_1}{\sqrt{1 - 2H_1 \cos a_1 + H_1^2}} = \frac{H_2 H_3 \sin a_2}{\sqrt{1 - 2H_3 \cos a_2 + H_3^2}}$$

$$\equiv H(a_2) = \frac{H_2 H_3 \sin(\theta - a_1 - a_3)}{\sqrt{1 - 2H_3 \cos(\theta - a_1 - a_3) + H_3^2}} \equiv F(a_1, a_3) \tag{23}$$

From Eqs. (7) and (9)

$$1 < H_1 \leq 2$$

$$0 < H_3 \leq 1$$

$$0 < H_2 < 1 \tag{24}$$

It follows from Eqs. (22) and (24) that the maximum value of

$$G(a_1) \equiv \frac{H_1 \sin a_1}{\sqrt{1 - 2H_1 \cos a_1 + H_1^2}}$$

is unity. Similarly, the maximum value of

$$H(\alpha_2) \equiv \frac{H_2H_3 \sin \alpha_2}{\sqrt{1 - 2H_3 \cos \alpha_2 + H_3^2}}$$

is

$$H_2H_3 = \sqrt{\frac{2}{x(1+x)}} < 1 \text{ (see Eq. 9)} \quad .$$

From Eqs. (18) through (21) (and the above discussion), the function $G(\alpha_1)$ increases monotonically from zero, when $\alpha_1 = 0$, to unity, when $\alpha_1 = \alpha_{1m}$. It then decreases monotonically with α_1 for $\alpha_1 > \alpha_{1m}$. The function $H(\alpha_2) = H(\theta - \alpha_1 - \alpha_3)$ increases monotonically with α_1 from $H(\theta - \alpha_3) \geq 0$, when $\alpha_1 = 0$, to its maximum value $\sqrt{2/x(1+x)} < 1$, when $\alpha_2 = \alpha_{2m}(\alpha_1 = \theta - \alpha_3 - \alpha_{2m})$. The function $H(\alpha_2)$ then decreases monotonically with α_1 from the maximum value, when $\alpha_1 = \theta - \alpha_3 - \alpha_{2m}$, to zero, when $\alpha_1 = \theta - \alpha_3(\alpha_2 = 0)$. The variation with α_1 of $G(\alpha_1)$ and $H(\alpha_2)$ is shown in Figure 2. It follows from the above that Eq. (23) (the first in equation (17)) must have a solution for $\alpha_1 \leq \alpha_{1m}$ (see Eq. (21)), where

$$\alpha_{1m} = \text{arc cos} \frac{1}{H_1} = \text{arc cos} \sqrt{\frac{1+x}{2x}} \quad , \tag{25}$$

as is illustrated in Figure 2. The possibility of additional solutions when $\alpha_1 > \alpha_{1m}$ is investigated below.

As noted above, $G(\alpha_1)$ decreases monotonically with α_1 for $\alpha_1 > \alpha_{1m}$, reaching its minimum when $\alpha_1 = \theta - \alpha_3$. It therefore follows that Eq. (23) can have

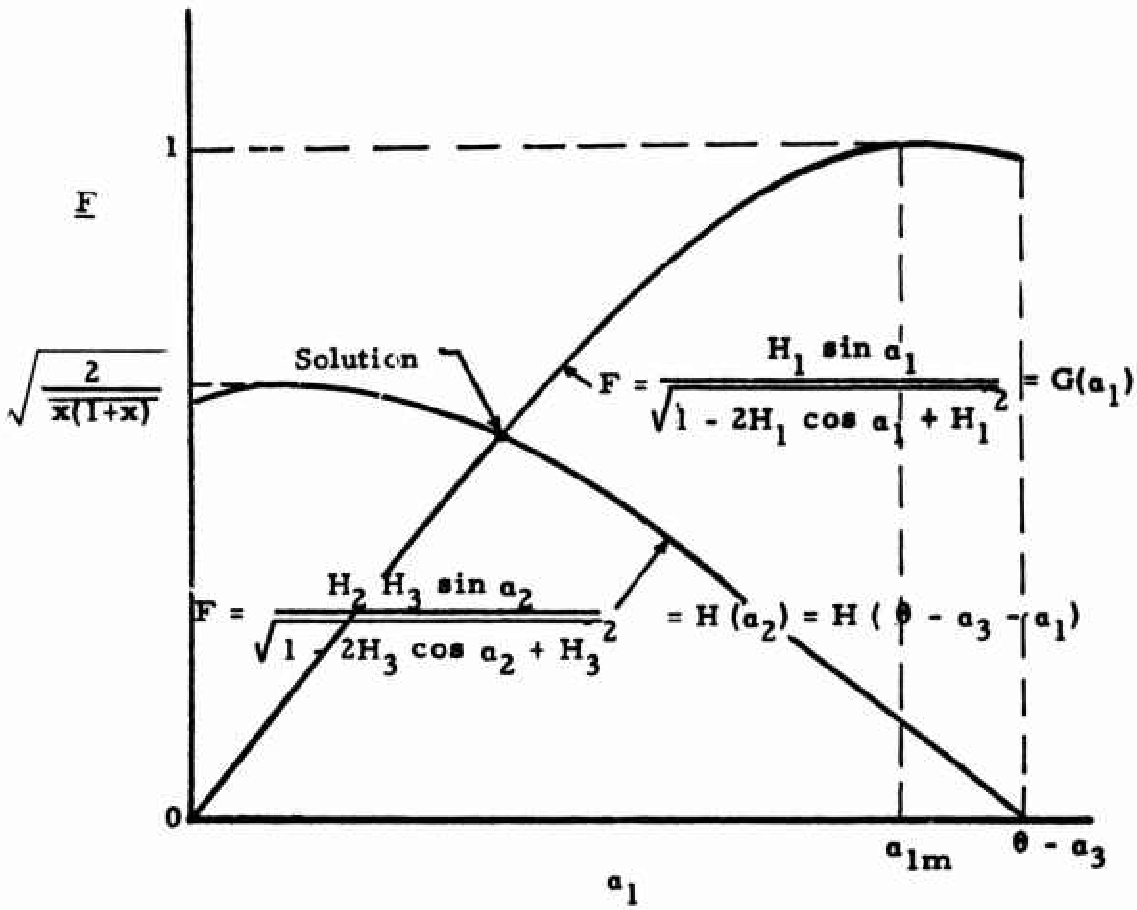

Figure 2. Solution of Equation (23)

no solution for $\alpha_1 > \alpha_{1m}$ if, (see Figure 2)

$$\min_{\alpha_1 > \alpha_{1m}} G(\alpha_1) = \frac{H_1 \sin(\theta - \alpha_3)}{\sqrt{1 - 2H_1\cos(\theta - \alpha_3) + H_1^2}} >$$

$$\max H(\alpha_2) = \max \frac{H_2H_3 \sin \alpha_2}{\sqrt{1 - 2H_3\cos \alpha_2 + H_3^2}}$$

$$= H_2H_3 = \sqrt{\frac{2}{x(1 + x)}} \quad , \tag{26}$$

where H_1 is a function of x given by Eq. (7). Equation (26) will be shown to be true over all but a narrow band of possible x, θ-values.

Equation (26) is true if, and only if

$$\theta - \alpha_3 < \arccos\left[\sqrt{\frac{2}{x^3(1 + x)}} - \sqrt{\frac{2 + x^3 + x^4 - x - 3x^2}{x^3(1 + x)}}\right]$$

$$= \arccos x^{-3/2}\left[\sqrt{\frac{2}{1 + x}} - (x - 1)\sqrt{x + 2}\right] \equiv \theta_m \quad . \tag{27}$$

In summary, it is impossible for Eq. (23) to have a solution for $\alpha_1 > \alpha_{1m}$, if Eq. (27) is satisfied. Since the present analysis is restricted to plane change angles in the range $\theta \leq 90°$, there can be <u>no</u> solutions of Eq. (23) for $\alpha_1 > \alpha_{1m}$ if

$$\cos \theta_m < 0 \tag{28}$$

Since Eq. (27) must be true if Eq. (28) applies.

From Eqs. (27) and (28), it follows that for any $\theta \leq 90°$, Eq. (23) cannot have a solution for $\alpha_1 > \alpha_{1m}$ if

$$f(x) \equiv x^3 + x^2 - 3x - 1 > 0 \quad . \tag{29}$$

Equation (29) is true if, and only if

$$x > \bar{x} \approx 1.48 \tag{30}$$

where $\bar{x}$ is the single positive root of the cubic equation $f(x) = 0$ (see Eq. 29). Furthermore, $x = r_t/r_i \geq r_f/r_i$.

Figure 3 is a graph of θ_m versus x. It will be assumed, without proof, that any solution $\alpha_1 = \alpha_{1s}$ of Eq. (23) which minimizes ΔV_T for a given value of α_3 satisfies the inequality $\alpha_{1s} < \alpha_{1m}$ in the narrow x, θ - region excluded from the above proof (that area above the θ_m versus x curve in Figure 3). Therefore, the values of $\alpha_1 = \alpha_{1s}$, which minimize ΔV_T for given values of x, θ, and α_3, are obtained by solving Eq. (23). The above α_{1s} values that satisfy Eq. (23) also satisfy the inequality $\alpha_{1s} \leq \alpha_{1m}$.

Equation (23) can be restated as

$$\cos \alpha_1 = \frac{1}{H_1}\left[S \pm \sqrt{S^2 - S\left(1 + H_1^2\right) + H_1^2}\right] \tag{31}$$

where $S = [F(\alpha_1, \alpha_3)]^2 = [H(\alpha_2)]^2$. The two choices of sign in Eq. (31) correspond to values of α_1 greater or less than α_{1m}. Since the present investigation is restricted to values of α_1 less than or equal to α_{1m}, only the positive sign need be considered. It, therefore, follows that

$$\cos \alpha_1 = \frac{1}{H_1}\left[S + \sqrt{S^2 - S\left(1 + H_1^2\right) + H_1^2}\right] \tag{32}$$

(cont.)

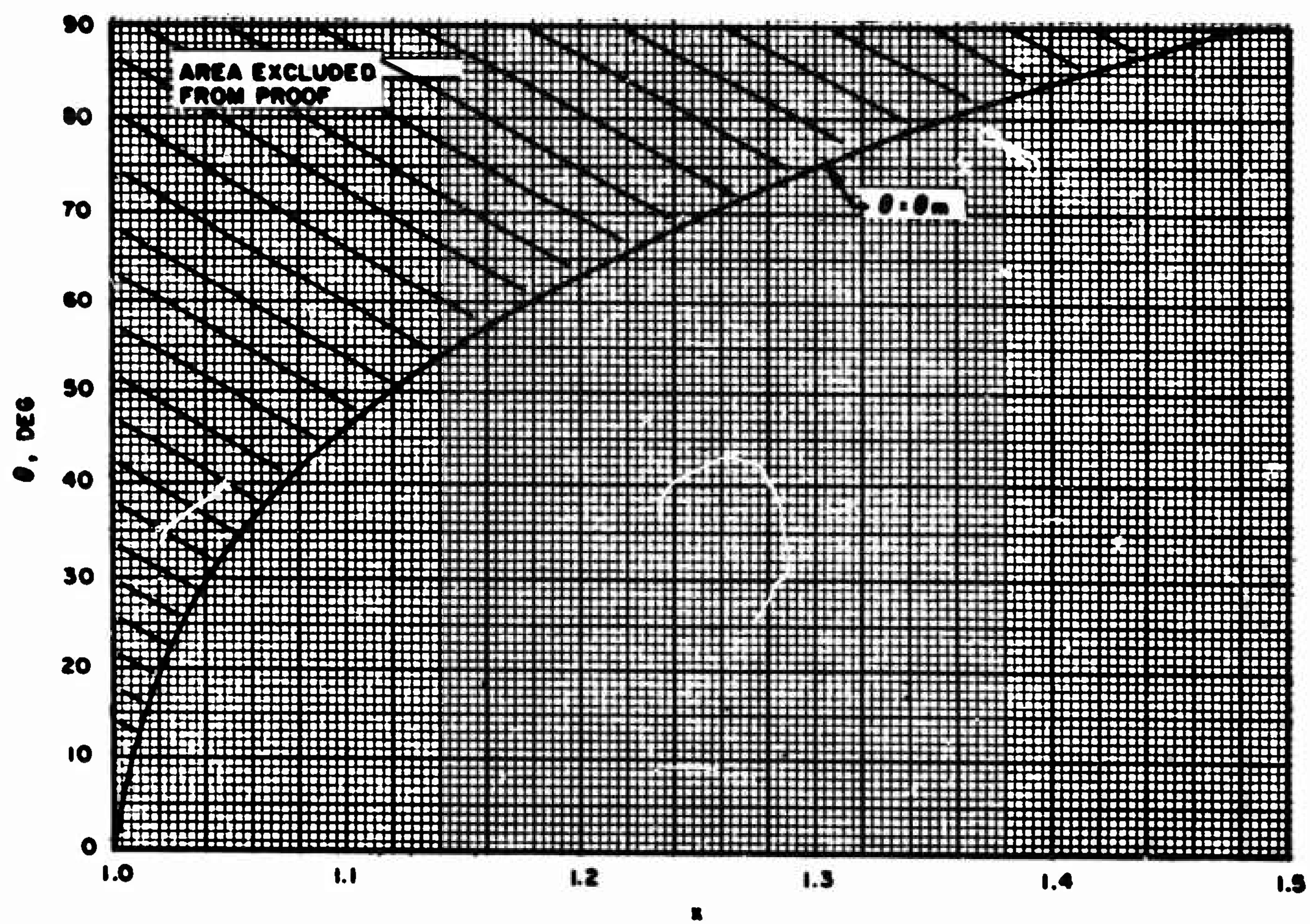

Figure 3. Values of x and θ Where Proof That $a_{1s} \leq a_{1m}$ Applies

$$\sin \alpha_1 = \frac{1}{H_1} F(\alpha_1, \alpha_3) \sqrt{1 - 2H_1 \cos \alpha_1 + H_1^2} \quad . \tag{32}$$

From Eqs. (18) through (22), α_1 is a monotonically increasing function of $F(\alpha_1, \alpha_3) = H(\alpha_2) = G(\alpha_1)$ in the interval $0 \le \alpha_1 \le \alpha_{1m}$. Furthermore, all solutions $\alpha_1 = \alpha_{1s}$ of Eq. (23) lie in the above interval. If $\bar{\alpha}_1$ is that value of α_1 at which

$$F(\alpha_1, \alpha_3) = \max H(\alpha_2) = H_2 H_3 = \sqrt{\frac{2}{x(1 + x)}} \quad ,$$

it follows from the above monoticity arguments that $\alpha_{1s} \le \bar{\alpha}_1$ for all x, y, and θ.

Since $\alpha_1 = \bar{\alpha}_1$ when $F(\alpha_1, \alpha_3) = H(\alpha_2) = H_2 H_3$, it follows from Eqs. (7), (9), and (32) that

$$\cos \bar{\alpha}_1 = \sqrt{\frac{2}{x^3(1 + x)}} + \frac{(x - 1)}{x} \sqrt{\frac{x + 2}{x}}$$

$$\sin \bar{\alpha}_1 = + \sqrt{1 - \cos^2 \bar{\alpha}_1} \quad . \tag{33}$$

Figure 4 is a graph of $\bar{\alpha}_1$ versus x. It is interesting to note that

$$\alpha_{1s} \le \bar{\alpha}_1 < K \approx 5.30^\circ \tag{34}$$

regardless of the choice of x, y, and θ. It can be shown that $\bar{\alpha}_1$ is maximized when $x = (1 + \sqrt{7})/2 \approx 1.82$.

The second equation of (14) (Eq. (13)) is treated below. A bound is similarly obtained on the angle $\alpha_3 = \alpha_{3s}$ which minimizes ΔV_T for a given value of α_1. Substitution of the second equation of (10) into Eq. (13) yields the algebraic equatio

$$\frac{H_4 H_5 \sin \alpha_3}{\sqrt{1 - 2H_5 \cos \alpha_3 + H_5^2}} = \frac{H_2 H_3 \sin \alpha_2}{\sqrt{1 - 2H_3 \cos \alpha_2 + H_3^2}} = F(\alpha_1, \alpha_3) \tag{35}$$

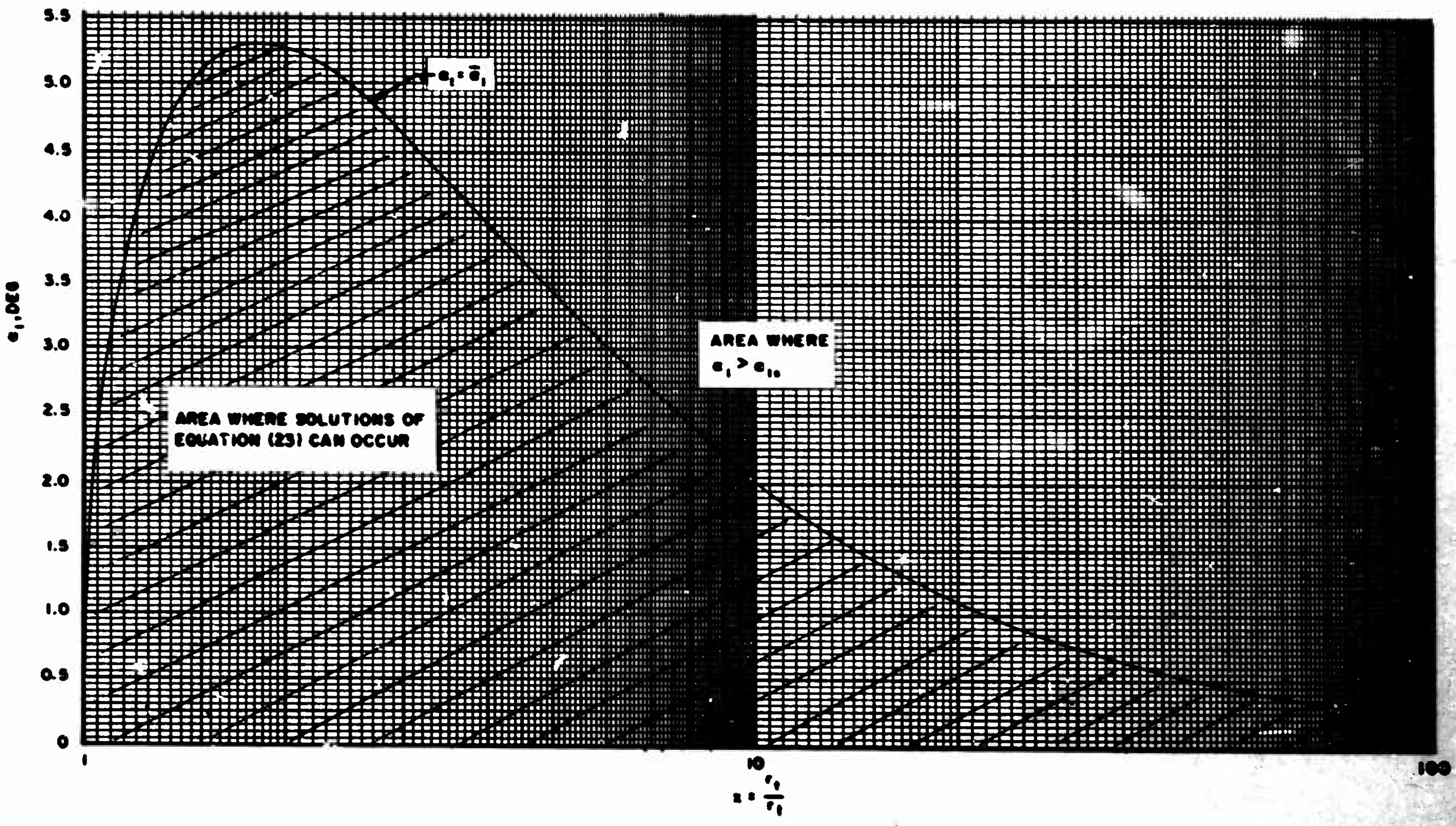

Figure 4. Bounding Curve for Possible Solutions of Equation (23)

Using the same line of reasoning as in the investigation of Eq. (23), it is readily shown that, for any α_1, there exists an $\alpha_3 < \alpha_{3m}$, which satisfies Eq. (35) where

$$\alpha_{3m} = \arc\cos \frac{1}{H_5} = \arc\cos \sqrt{\frac{1+y}{2y}} \quad . \tag{36}$$

It can also be shown that Eq. (35) cannot have a solution for $\alpha_3 > \alpha_{3m}$ if

$$\frac{H_4 H_5}{\sqrt{1 + H_5^2}} = \sqrt{\frac{2y^2}{x(1+3y)}} > H_2 H_3 = \sqrt{\frac{2}{x(1+x)}}$$

or

$$\frac{y^2}{1+3y} > \frac{1}{1+x} \tag{37}$$

The inequality in Eq. (34) is true if, and only if

$$y > \bar{y} \tag{38}$$

where $\bar{y}$ is the single positive root of the cubic equation

$$\left(\frac{x}{y}\right)\bar{y}^3 + \bar{y}^2 - 3\bar{y} - 1 = 0 \tag{39}$$

Figure 5 is a graph of $\bar{y}$ versus $x/y = r_f/r_i$.

It follows from Eq. (39) that $\bar{y} < 1$ for $r_f/r_i > 3$. Since $y = r_t/r_f \geq 1$, Eq. (37) is always true when $x/y = r_f/r_i > 3$. Furthermore, it is shown above that Eq. (35) cannot have a solution for $\alpha_3 > \alpha_{3m}$ when Eq. (37) is true. The remaining investigation is therefore concerned with values of r_f/r_i in the range $1 \leq r_f/r_i \leq 3$.

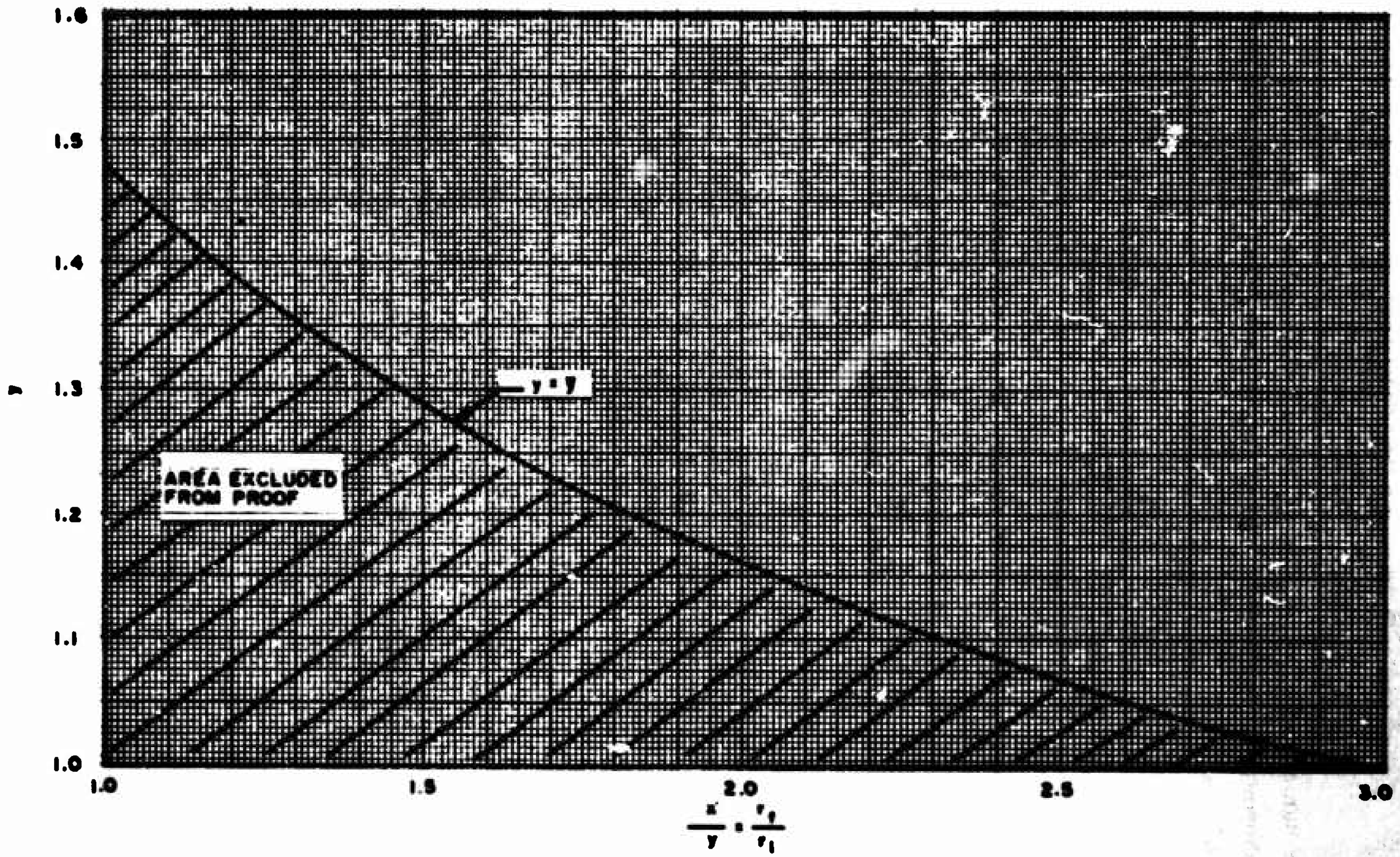

Figure 5. Variation of $\bar{\gamma}$ with r_f/r_i

For $y \le \bar{y}$ (see Eq. (38)), Eq. (35) cannot have a solution for $\alpha_3 > \alpha_{3m}$ if

$$\frac{H_4H_5\sin(\theta - \alpha_1)}{\sqrt{1 - 2H_5\cos(\theta - \alpha_1) + H_5^2}} > \max \frac{H_2H_3\sin \alpha_2}{\sqrt{1 - 2H_3\cos \alpha_2 + H_3^2}} = \sqrt{\frac{2}{x(1 + x)}} \qquad (40)$$

The inequality in Eq. (40) is true if and only if (see Eqs. (26) and (27))

$$\theta - \alpha_1 < \theta'_m \equiv \arccos\left[\sqrt{\frac{2(1 + y)}{y^3(1 + uy)^2}} - \sqrt{\frac{(1 + uy)^2y^3 + 2(1 + y) - (1 + uy)(1 + 3y)y}{y^3(1 + uy)^2}}\right] \qquad (41)$$

where $u = x/y = r_f/r_i$. Figure 6 is a graph of θ'_m versus r_f/r_i for various values of y.

It will be assumed that $\alpha_{3s} \le \alpha_{3m}$ for all values of x, y, and θ, as it was previously assumed that $\alpha_{1s} < \alpha_{1m}$ for all x and θ.

By analogy to the transformation of Eq. (23) into Eq. (32), Eq. (35) can be written as

$$\cos \alpha_3 = \frac{1}{H_4^2H_5}\left[S + \sqrt{S^2 - SH_4^2\left(1 + H_5^2\right) + H_4^4H_5^2}\right]$$

$$\sin \alpha_3 = \frac{F(\alpha_1, \alpha_3)}{H_4H_5}\sqrt{1 - 2H_5\cos \alpha_3 + H_5^2} \qquad (42)$$

where $S = [F(\alpha_1, \alpha_3)]^2$.

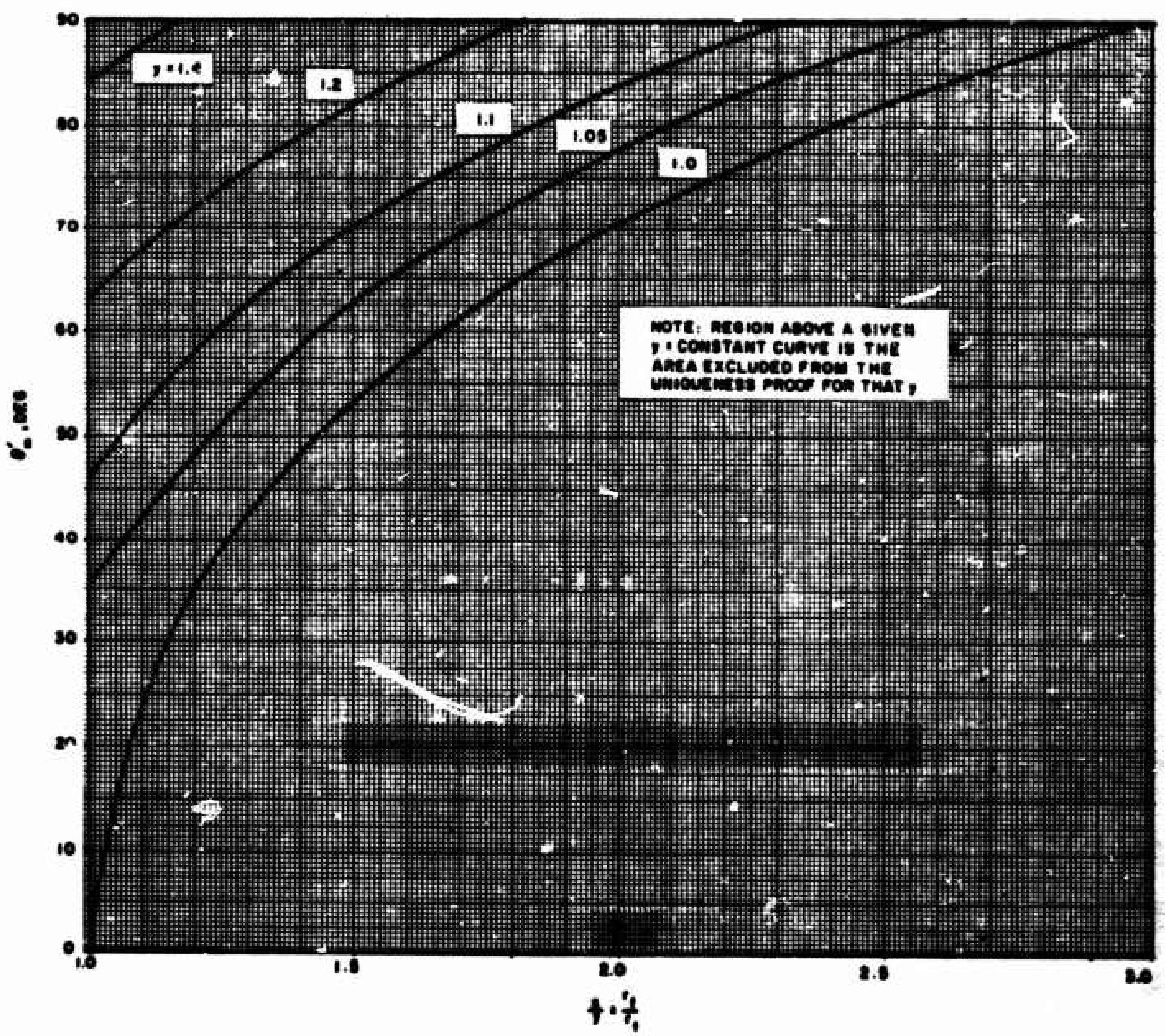

Figure 6. Bounding Curves for a_3 Uniqueness Proof

From Eqs. (18) through (22), α_3 is a monotonically increasing function of

$$F(\alpha_1, \alpha_3) = H(\alpha_2) = \frac{H_4 H_5 \sin \alpha_3}{\sqrt{1 - 2H_5 \cos \alpha_3 + H_5^2}}$$

(see Eq. (35)) in the interval $0 \leq \alpha_3 \leq \alpha_{3m}$. Furthermore, all solutions $\alpha_3 = \alpha_{3_s}$ of Eq. (35)(or, equivalently, Eq. (42)) lie in the above interval. If $\bar{\alpha}_3$ is that value of α_3 at which

$$F(\alpha_1, \alpha_3) = \max F(\alpha_1, \alpha_3) = \sqrt{\frac{2}{x(1 + x)}} \quad ,$$

then it follows from the above monoticity property that $\alpha_{3_s} \leq \bar{\alpha}_3$ for all x, y, and θ.

Since $\alpha_3 = \bar{\alpha}_3$ when

$$F(\alpha_1, \alpha_3) = \max F(\alpha_1, \alpha_3) = \sqrt{\frac{2}{x(1 - x)}} \quad ,$$

it follows from Eqs. (9), (11), and (42) that

$$\cos \bar{\alpha}_3 = \sqrt{\frac{2(1 + y)}{y^3(1 + uy)^2}} + \sqrt{\frac{(1 + uy)^2 y^3 + 2(1 + y) - (1 + uy)(1 + 3y)y}{y^3(1 + uy)^2}}$$

$$\sin \bar{\alpha}_3 = \sqrt{1 - \cos^2 \bar{\alpha}_3} \quad , \tag{43}$$

where $u = x/y = r_f/r_i$. Figure 7 is a graph of $\bar{\alpha}_3$ versus y for various values of r_f/r_i.

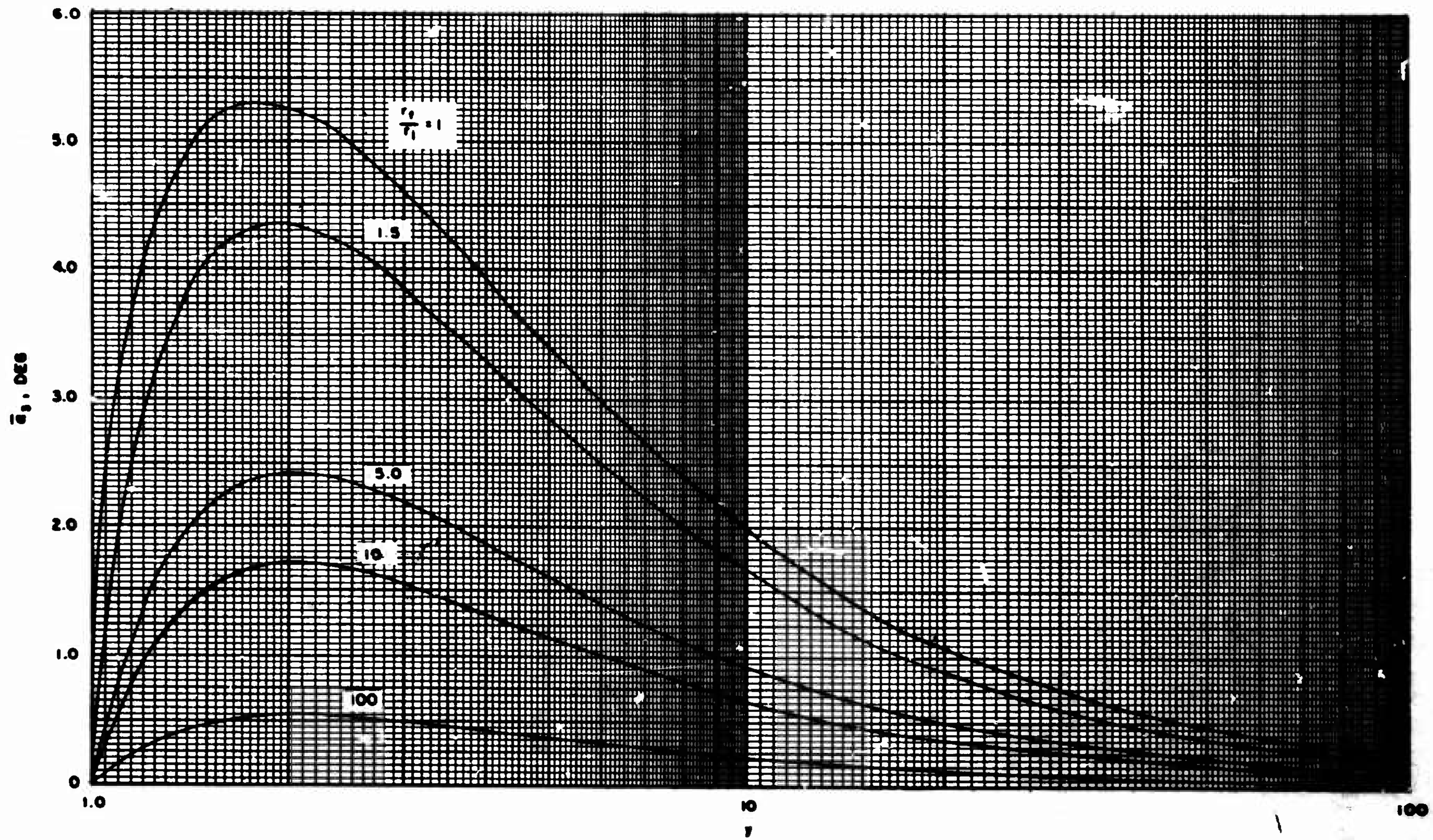

Figure 7. Bounding Curve for Solutions of Equation (35)

It is interesting to note (see Eq. (34)) that

$$\alpha_{3s} \leq \bar{\alpha}_3 < K \approx 5.30° \qquad (44$$

for any choice of x, y, and θ. Comparison of Eqs. (33) and (43) shows that $\bar{\alpha}_1 = \bar{\alpha}_3$, when y = x. Furthermore, the maximum value of $\bar{\alpha}_3$ occurs when $x/y = r_f/r_i = 1$.

From Eqs. (33) and (43)

$$\alpha_{3s} \rightarrow 0 \text{ as } y \rightarrow 1$$

$$\alpha_{1s} \rightarrow 0 \text{ as } x \rightarrow 1 \qquad (45)$$

Also, since x ≥ y, as x → 1, y → 1 and thus

$$\alpha_{3s} \rightarrow 0 \text{ as } x \rightarrow 1 \quad . \qquad (46)$$

VI. TRANSFER ALTITUDE h_t BETWEEN INITIAL ALTITUDE h_i AND FINAL ALTITUDE h_f

The analysis of the bi-elliptic transfer, with $r_t \geq r_f$, treated in the previous sections, was restricted to the case where $r_i \neq r_f$ (except for the degenerate case $r_i = r_f = r_t$, where $a_{1s} = a_{3s} = 0$). Consideration will now be given to the case where $r_i \leq r_t < r_f$.

The velocity ΔV_T required to transfer from altitude h_i to h_t to h_f with plane change θ is given by Eqs. (6) through (12), where ΔV_1 is the velocity increment at $h = h_i$, ΔV_2 is the increment at $h = h_t$, and ΔV_3 is the increment at $h = h_f$. It thus appears that there is no difference between the cases $r_i \leq r_f \leq r_t$ and $r_i \leq r_t < r_f$. It can, in fact, be shown that when $r_i \leq r_t < r_f$, the minimum value of ΔV_T can be obtained by solving Eq. (17).

However, when $r_i \leq r_t < r_f$, $y < 1$ and, therefore, the various bounding theorems such as Eqs. (33) and (43) no longer apply. It is also convenient, when employing iterative methods, to utilize independent variables that can be bounded in the manner a_1 and a_3 were bounded in the first problem ($0 \leq a_1 \leq a_{1m}$, $0 \leq a_3 \leq a_{3m}$). Bounds will therefore be sought for two of the three variables a_1, a_2, and a_3, and the existence of a minimum will be established. Due to the length of the necessary exposition, uniqueness will not be demonstrated. The uniqueness proof is of essentially the same nature as the one for the case $r_i \leq r_f \leq r_t$ in Reference 5.

Suppose that the two independent variables are chosen as a_1 and a_3. It then follows that the two partial derivatives of ΔV_T are given by Eq. (13). If $x \neq 1$ then the conclusions in Eq. (14) also apply.

From Eqs. (7), (18), (21), and (22), the function

$$G(a_1) \equiv \frac{H_1 \sin a_1}{\sqrt{1 - 2H_1 \cos a_1 + H_1^2}} \tag{47}$$

increases from zero when $\alpha_1 = 0$, to unity when

$$\alpha_1 = \alpha_{1m} = \arccos\sqrt{\frac{1 + x}{2x}}$$

(see Eq. (25)).

Furthermore, $G(\alpha_1)$ decreases as α_1 increases for $\alpha_1 > \alpha_{1m}$. From Eqs. (9), (18), (21), and (22), the function

$$H(\alpha_2) = H(\theta - \alpha_1 - \alpha_3) = \frac{H_2 H_3 \sin \alpha_2}{\sqrt{1 - 2H_3 \cos \alpha_2 + H_3^2}} \tag{48}$$

increases from

$$\frac{H_2 H_3 \sin(\theta - \alpha_3)}{\sqrt{1 - 2H_3 \cos(\theta - \alpha_3) + H_3^2}}$$

when $\alpha_1 = 0$, to

$$\max H(\alpha_2) = \sqrt{\frac{2}{x(1 + x)}} < 1 \tag{49}$$

when

$$\cos(\theta - \alpha_1 - \alpha_3) = \sqrt{\frac{1 + y}{1 + x}} \equiv \cos \alpha_{2m} \quad .$$

The function $H(\alpha_2)$ decreases monotonically with α_1 from its maximum value to zero when $\alpha_1 = \theta - \alpha_3$.

From Eqs. (13), (47), (48), and (49), there exists an $\alpha_1 = \alpha_{1s} < \alpha_{1m}$ at which $\partial \Delta V_T / \partial \alpha_1 = 0$.

It is readily shown that $\alpha_{1s} \leq \bar{\alpha}_1$ (see Eq. (33)) and, therefore, all solutions $\alpha_1 = \alpha_{1s}$ of Eq. (23) lie below the curve $\alpha_1 = \bar{\alpha}_1$ in Figure 4.

From Eqs. (11), (18), and (22) (also see Eq. (10))

$$\max K(\alpha_3) \equiv \max \frac{H_4 H_5 \sin \alpha_3}{\sqrt{1 - 2H_5 \cos \alpha_3 + H_5^2}}$$

$$= y\sqrt{\frac{2}{x(1+y)}} \quad . \tag{50}$$

It can be shown, employing Eqs. (47) through (50), that the equation $\partial \Delta V_T / \partial \alpha_3 =$ must have a solution

$$\alpha_3 = \alpha_{3s} < \alpha_{3m} = \arccos \sqrt{\frac{2y}{1+y}} \quad ,$$

(see Eqs. (13), (23), (49), and (50)) if

$$y\sqrt{\frac{2}{x(1+y)}} \geq \sqrt{\frac{2}{x(1+x)}}$$

or

$$x \leq \frac{1+y}{y^2} - 1 \quad . \tag{51}$$

If Eq. (51) is true, then the methods of proving the existence of, and obtaining solutions $\alpha_1 = \alpha_{1s}$ and $\alpha_3 = \alpha_{3s}$ to Eqs. (15) and (16) (yielding the minimum value of ΔV_T for given x, y, and θ) are identical to the methods utilized for the case $h_i \leq h_f \leq h_t$ previously analyzed.

Regardless of whether Eq. (51) applies, $\alpha_1 \le \bar{\alpha}_1 \to 0$ as $x \to 1$. It can be inferred from continuity arguments or demonstrated directly that α_1 assumes its limiting value $\alpha_1 = 0$, when $x = 1$. It was previously shown that $\alpha_3 = 0$ when $y = 1$ (see Eq. (45)). From the two limiting cases ($\alpha_1 = 0$ when $x = 1$, $\alpha_3 = 0$ when $y = 1$), it is concluded that, for a Hohmann transfer with plane change, no plane change should be made prior to entering or after leaving the transfer ellipse. Furthermore, since the Hohmann transfer, $x = 1$ or $y = 1$, was previously treated, the present investigation can be limited to the region $h_i < h_t < h_f$.

Because convenient bounds cannot always be placed on the variable α_3, consideration will be given to the independent variable pair α_1 and α_2. The partials of ΔV_T with respect to α_1 and α_2 are given by the expressions (see Eqs. (6) through (12)):

$$\frac{1}{V_{ci}} \frac{\partial \Delta V_T}{\partial \alpha_1} = \frac{H_1 \sin \alpha_1}{\sqrt{1 - 2H_1 \cos \alpha_1 + H_1^2}} - \frac{H_4 H_5 \sin \alpha_3}{\sqrt{1 - 2H_5 \cos \alpha_3 + H_5^2}}$$

$$\frac{1}{V_{ci}} \frac{\partial \Delta V_T}{\partial \alpha_2} = \frac{H_2 H_3 \sin \alpha_2}{\sqrt{1 - 2H_3 \cos \alpha_2 + H_3^2}} - \frac{H_4 H_5 \sin \alpha_3}{\sqrt{1 - 2H_5 \cos \alpha_3 + H_5^2}} \tag{52}$$

From Eqs. (47) through (52), it follows that (see Eq. (51)), when

$$x > \frac{1 + y}{y^2} - 1 \tag{53}$$

there exist an $\alpha_1 = \alpha_{1s} < \alpha_{1m}$ and $\alpha_2 = \alpha_{2s} < \alpha_{2m}$, which respectively satisfy

$$\frac{\partial \Delta V_T}{\partial \alpha_1} = 0 \text{ for any } \alpha_2 \tag{54}$$

(cont.)

$$\frac{\partial \Delta V_T}{\partial \alpha_2} = 0 \text{ for any } \alpha_1 \tag{54}$$

In summary, it is seen that two angles are bounded regardless of the values of x and y. If Eq. (51) applies, then $\alpha_{1s} < \alpha_{1m}$ and $\alpha_{3s} < \alpha_{3m}$. If Eq. (53) is true (and, therefore, Eq. (51) is false), then $\alpha_{1s} < \alpha_{1m}$ and $\alpha_{2s} < \alpha_{2m}$. In either case, $\alpha_{1s} \leq \bar{\alpha}_1$, where $\bar{\alpha}_1$ is defined in Eq. (33) and depicted in Figure 4.

It can be shown that the two equations in (54) have a single solution $\alpha_1 = \alpha_{1s}$, $\alpha_2 = \alpha_{2s}$ when the inequality in (53) is true. Since $\alpha_{3s} = \theta - \alpha_{1s} - \alpha_{2s}$, and sinc the two equations in (54) are equivalent to the two equations in (15) and (16), it follows from the above discussion and the discussion subsequent to Eq. (51) tha Eq. (54) has a single solution $\alpha_1 = \alpha_{1s}$, $\alpha_3 = \alpha_{3s}$ for all x and y when $h_i < h_t < h_f$ It was previously demonstrated that the solution obtained yields the minimum value of ΔV_T, when Eq. (53) is false (see Eqs. (14) and (51). It is demonstrated below that the solution also yields the minimum value of ΔV_T when Eq. (53) is tr

From Eq. (52)

$$\frac{\partial \Delta V_T}{\partial \alpha_1} < 0 \text{ when } \alpha_1 = 0$$

$$\frac{\partial \Delta V_T}{\partial \alpha_1} > 0 \text{ when } \alpha_1 = \theta - \alpha_2$$

$$\frac{\partial \Delta V_T}{\partial \alpha_2} < 0 \text{ when } \alpha_2 = 0$$

$$\frac{\partial \Delta V_T}{\partial \alpha_2} > 0 \text{ when } \alpha_2 = \theta - \alpha_1 \tag{55}$$

It follows from Eq. (55) that the single solution of Eq. (54) corresponds to the minimum value of ΔV_T for $h_i < h_t < h_f$.

When Eq. (53) is true ($a_1 \leq a_{1m}$, $a_2 \leq a_{2m}$), then Eqs. (52) and (54) can be combined to yield the following equations:

$$\cos a_1 = \frac{1}{H_1}\left[S + \sqrt{S^2 - S\left(1 + H_1^2\right) + H_1^2}\right]$$

$$\sin a_1 = \frac{1}{H_1} F(a_1, a_2) \sqrt{1 - 2H_1 \cos a_1 + H_1^2}$$

$$\cos a_2 = \frac{1}{H_2^2 H_3}\left[S + \sqrt{S^2 - SH_2^2\left(1 + H_3^2\right) + H_2^4 H_3^2}\right]$$

$$\sin a_2 = \frac{F(a_1, a_2)}{H_2 H_3} \sqrt{1 - 2H_3 \cos a_2 + H_3^2} \tag{56}$$

where $S = [F(a_1, a_2)]^2 = [K(a_3)]^2$.

VII. TRANSFER ALTITUDE h_t LESS THAN INITIAL ALTITUDE h_i

In the previous sections, the bi-elliptic transfer has been analyzed in the region $h_f \le h_t$ and $h_i \le h_t < h_f$. It therefore follows from Eq. (1) that the only region to be investigated is $h_t < h_i$. As previously noted, the velocity increments for any values of h_i, h_t, and h_f are given by Eqs. (6) through (12); the two partial derivatives of interest are given by (52). The conclusions in Eq. (55) also apply in the present region.

In Section V bounds were obtained for the two independent variables α_1 and α_3. Similar bounds will be obtained in this section for the variables α_1 and α_2

In the region $h_t < h_i < h_f$ (see Eqs. (7), (9), and (11))

$$H_1 < 1 \quad , \quad H_3 < 1 \quad , \quad H_5 < 1 \tag{57}$$

From Eqs. (7) through (11), (18) through (22), (47) through (50), and (57), it follows that:

a. The function $G(\alpha_1)$ increases monotonically from zero when $\alpha_1 = 0$, to $G(\alpha_1) = H_1$ when $\alpha_1 = \alpha_{1m} = \text{arc cos } H_1$. For $\alpha_1 > \alpha_{1m}$, $G(\alpha_1)$ decreases monotonically as α_1 increases.

b. For a given value of α_2, the function $K(\alpha_3)$ increases monotonically from a value $K' > 0$ when $\alpha_1 = 0$, to $K(\alpha_3) = H_4H_5$ when $\alpha_3 = \alpha_{3m} = \theta - \alpha_1 - \alpha_2 = \text{arc cos } H_3$. The function $K(\alpha_3)$ decreases monotonically from its maximum, when $\alpha_1 = \theta - \alpha_2 - \alpha_{3m}$, to zero when $\alpha_1 = \theta - \alpha_2$.

c. For a given value of α_1, the function $K(\alpha_3)$ increases from $K(\alpha_3) = K'' > 0$ when $\alpha_2 = 0$, to $K(\alpha_3) = H_4H_5$ when $\alpha_2 = \theta - \alpha_1 - \alpha_{3m}$. It then decreases to zero when $\alpha_2 = \theta - \alpha_1$.

d. The function $H(a_2)$ increases monotonically from zero when $a_2 = 0$, to $H(a_2) = H_2H_3$ when $a_2 = a_{2m} = \arccos H_3$. For $a_2 > a_{2m}$, $H(a_2)$ decreases monotonically as a_2 increases.

From (a) through (d) and Eqs. (7), (9), and (11), it follows that for $h_t < h_i < h_f (x > y)$

$$\max K(a_3) = H_4H_5 = y\sqrt{\frac{2}{x(1+y)}} = \sqrt{\frac{y}{x}}\sqrt{\frac{2}{1+\frac{1}{y}}} < \sqrt{\frac{2}{1+\frac{1}{y}}} <$$

$$\max G(a_1) = \sqrt{\frac{2}{1+\frac{1}{x}}} = \sqrt{\frac{2x}{1+x}} < \max H(a_2) = \sqrt{\frac{2}{x(1+x)}} \tag{58}$$

From Eqs. (52) and (58), it follows that, for any a_2, there exists an $a_1 = a_{1s} < a_{1m}$ at which (see (a) through (d) above)

$$G(a_1) = K(a_3)$$

or

$$\frac{\partial \Delta V_T}{\partial a_1} = 0 \tag{59}$$

Similarly, for any a_1, there exists an $a_2 = a_{2s} < a_{2m}$ at which

$$H(a_2) = K(a_3)$$

or

$$\frac{\partial \Delta V_T}{\partial a_2} = 0 \tag{60}$$

From Eq. (55), there exists an $\alpha_{1s} < \alpha_{1m}$ ($\alpha_{2s} < \alpha_{2m}$), which satisfies Eq. (59) (Eq. (60)) and yields a relative minimum of $\Delta V_T(\alpha_1, \alpha_2)$. It can further be sho that only one value of $\alpha_1 = \alpha_{1s}$ satisfies Eq. (59) for each value of a α_2 and, si larly, for each value of α_1, there is only one $\alpha_2 = \alpha_{2s}$ that satisfies Eq. (60).

It can be shown (see Reference 5) that the simultaneous solution of Eqs. (59) and (60) yields the values of $\alpha_1 = \alpha_{1s}$ and $\alpha_2 = \alpha_{2s}$ which minimize ΔV_T where $\alpha_{1s} \le \alpha_{1m}$ and $\alpha_{2s} \le \alpha_{2m}$. Equations (59) and (60) can be written in the more explicit form shown in (56).

From Eq. (58) and the properties (a) through (d) preceding it, the function $H(\alpha_2)$ (Eq. (48)) increases from zero when $\alpha_2 = 0$, to $\max H(\alpha_2)$ when $\alpha_2 = \alpha_{2m}$. Furthermore, $\alpha_{2s} \le \alpha_{2m}$ is the solution of Eq. (60). If $\alpha_2' \le \alpha_{2m}$ is the value of α_2 for which (see Eq. (60)),

$$H(\alpha_2) = H(\alpha_2') = \max K(\alpha_3) \equiv \overline{K} = H_4 H_5 \tag{61}$$

and it follows from Eq. (58) that $\alpha_{2s} \le \alpha_2'$ for all x, y, and θ. From Eqs. (9), (11), and the third and fourth equations of (56), α_2' is given by

$$\cos \alpha_2' = \sqrt{\frac{1 + x}{u^3(x + u)}} + \sqrt{\frac{(u - x)(u^2 + xu - x^3 - x^2)}{u^3}}$$

$$\sin \alpha_2' = \sqrt{1 - \cos^2 \alpha_2'} \quad , \tag{62}$$

where $u = x/y = r_f/r_i$. Figure 8 is a graph of α_2' versus r_f/r_i for various values of x.

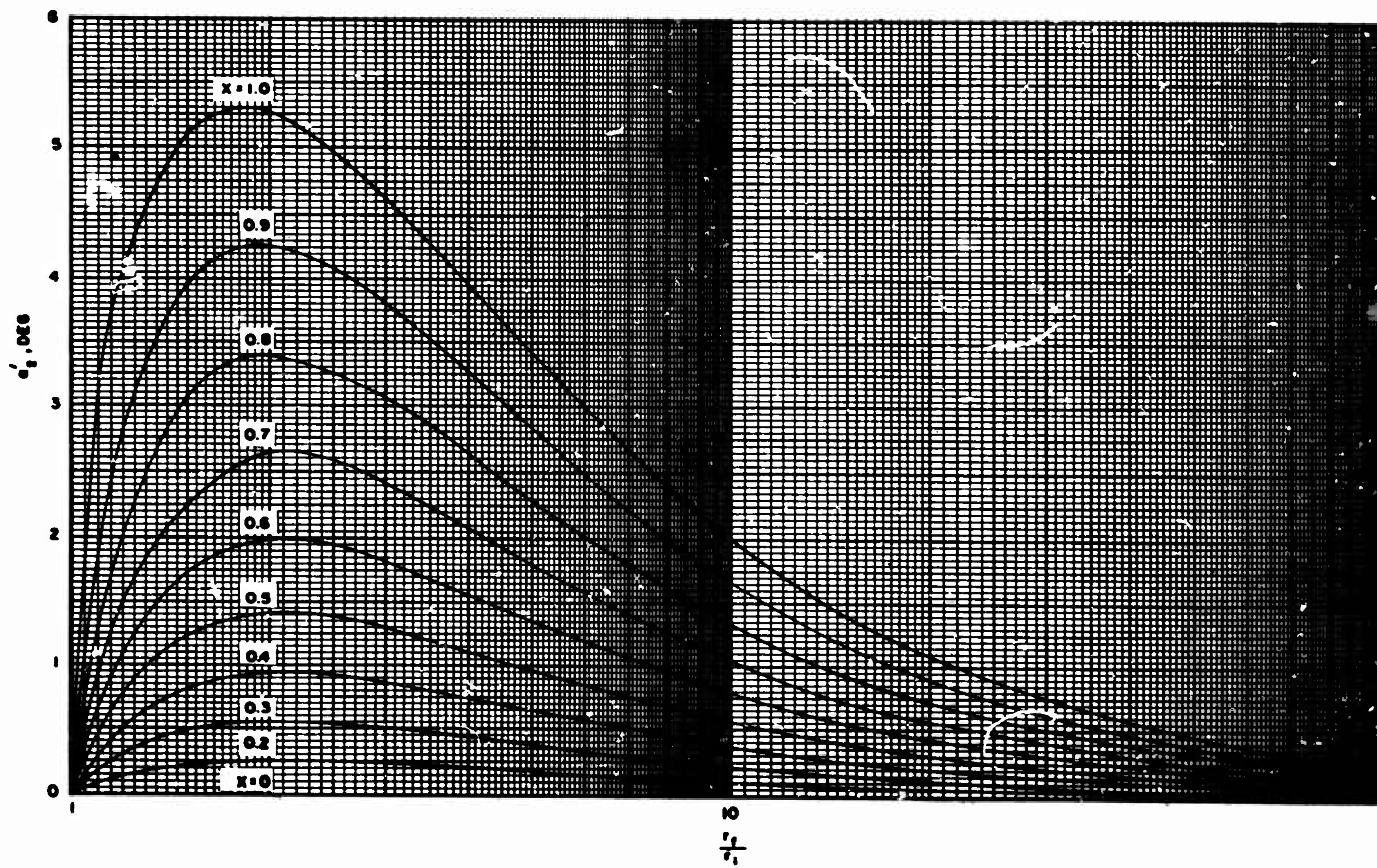

Figure 8. Bounding Curve for Solutions of Equation (60)

From Eq. (62)

a. $\alpha_{2s} \leq \alpha_2' \to 0$ as $x \to 0$

b. $\alpha_{2s} \leq \alpha_2' \to 0$ as $y \to x (u \to 1)$

c. For any $u = \frac{r_f}{r_i} > 1$, α_2' is maximized for $x = 1$.

It can be shown that α_{2s} assumes its limiting values $\alpha_{2s} = 0$ when $y = x$ or $x = 0$.

From Eq. (62) it follows that, when $x = 1$,

$$\cos \alpha_2' = \sqrt{\frac{2}{u^3(1 + u)}} + \frac{(u - 1)}{u} \sqrt{\frac{u + 2}{u}}$$

$$\sin \alpha_2' = \sqrt{1 - \cos^2 \alpha_2'} \tag{63}$$

Comparison of Eqs. (33) and (63) shows that α_2' is the same function of u at $x = 1$, as $\bar{\alpha}_1$ is of x. It thus follows from Eq. (34) (also see Eq. (44)) that

$$\alpha_{2s} \leq \alpha_2' < K \approx 5.30° \tag{64}$$

A bound can be placed on the angle α_{1s}, which is similar to the above bound on α_{2s}. From Eqs. (56) and (58), $\alpha_{1s} \leq \alpha_1' \leq \alpha_{1m}$, where α_1' is defined by the following equation (see Eq. (61)):

$$G(\alpha_1') = \max K(\alpha_3) = H_4 H_5$$

or

$$\cos a_1' = \sqrt{\frac{2x(1+x)}{u^2(u+x)^2}} + \sqrt{(u-x)(u^2+xu-x^3-x^2)}$$

$$\sin a_1' = \sqrt{1 - \cos^2 a_1'} \quad . \tag{65}$$

Figure 9 is a graph of a_1' versus x for various values of $u = r_f/r_i$.

From Eq. (65) and the previous analysis, it is clear that:

a. $a_{1s} \le a_1' \le a_{1m}$

b. $a_1' = a_{1m} = \sqrt{\frac{2x}{1+x}}$ when $u = 1$

c. $a_1' \rightarrow 0$ as $u \rightarrow \infty$

d. $a_1' = 0$ when $x = 1$

e. $a_1' = \arcsin \frac{1}{u} = \arcsin \frac{r_i}{r_f}$ when $x = 0$.

Figures 10 through 16 were prepared by Mr. Jerome Baker using the results of a computer program based on the technique developed in this report. Figures 10 through 12 show the minimum bi-elliptic velocity increment for various values of $r_t \ge r_i$, i.e., for intermediate and outer bi-elliptic transfers (see Eq. (1)).

The plane change angles θ, corresponding to Figures 10 through 12, are 0°, 10°, 20°, and 30°, respectively. There is evidently no minimization involved for the $\theta = 0°$ case which is treated in Reference 1.

An alternative three-dimensional transfer procedure is the modified Hohmann transfer described in Reference 2. The velocity requirement for this transfer

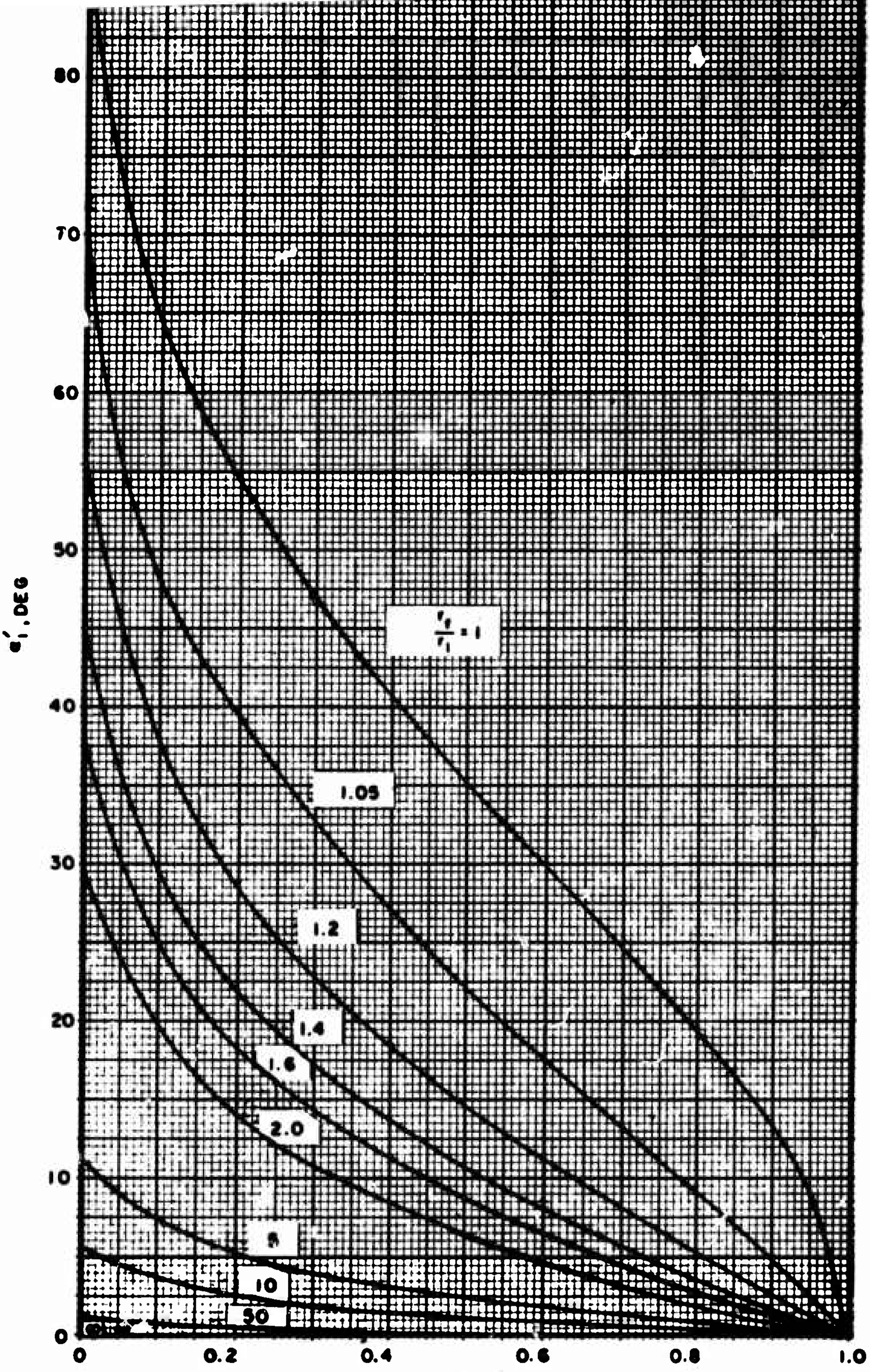

Figure 9. Bounding Curve for Solutions of Equation (59)

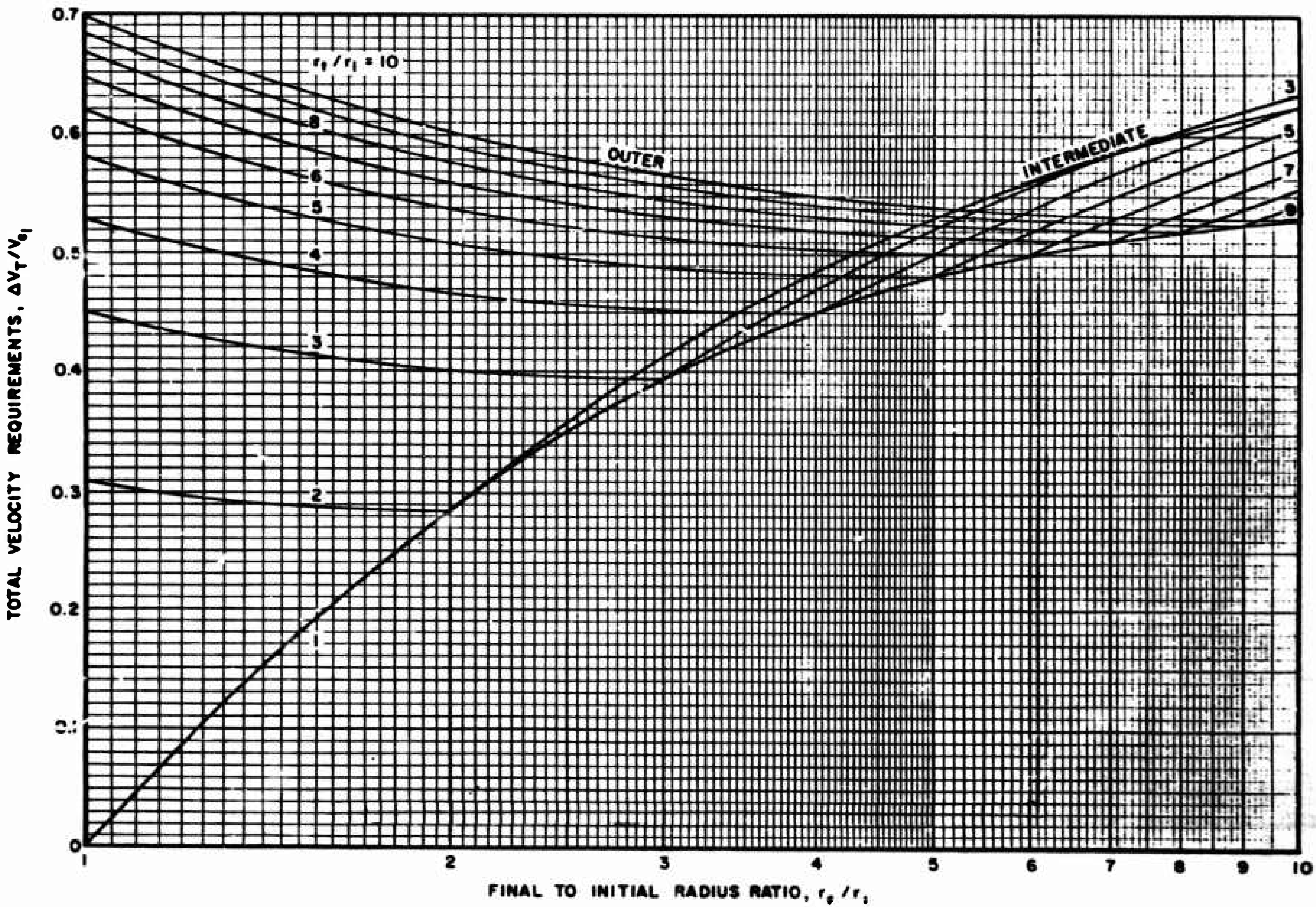

Figure 10. Minimum Bi-elliptic Velocity Requirements When $\theta = 0°$

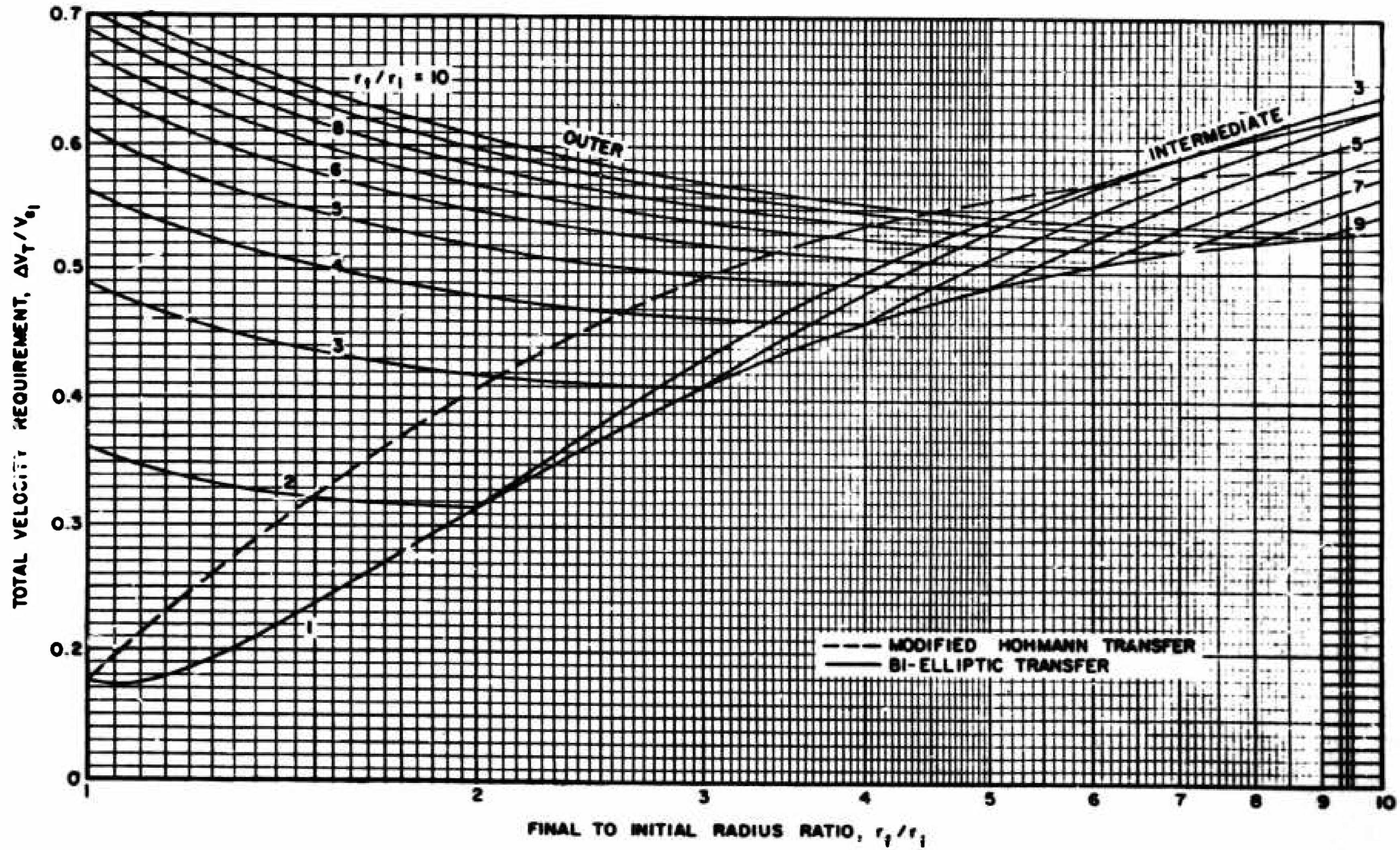

Figure 11. Minimum Bi-elliptic Velocity Requirements When $\theta = 10°$

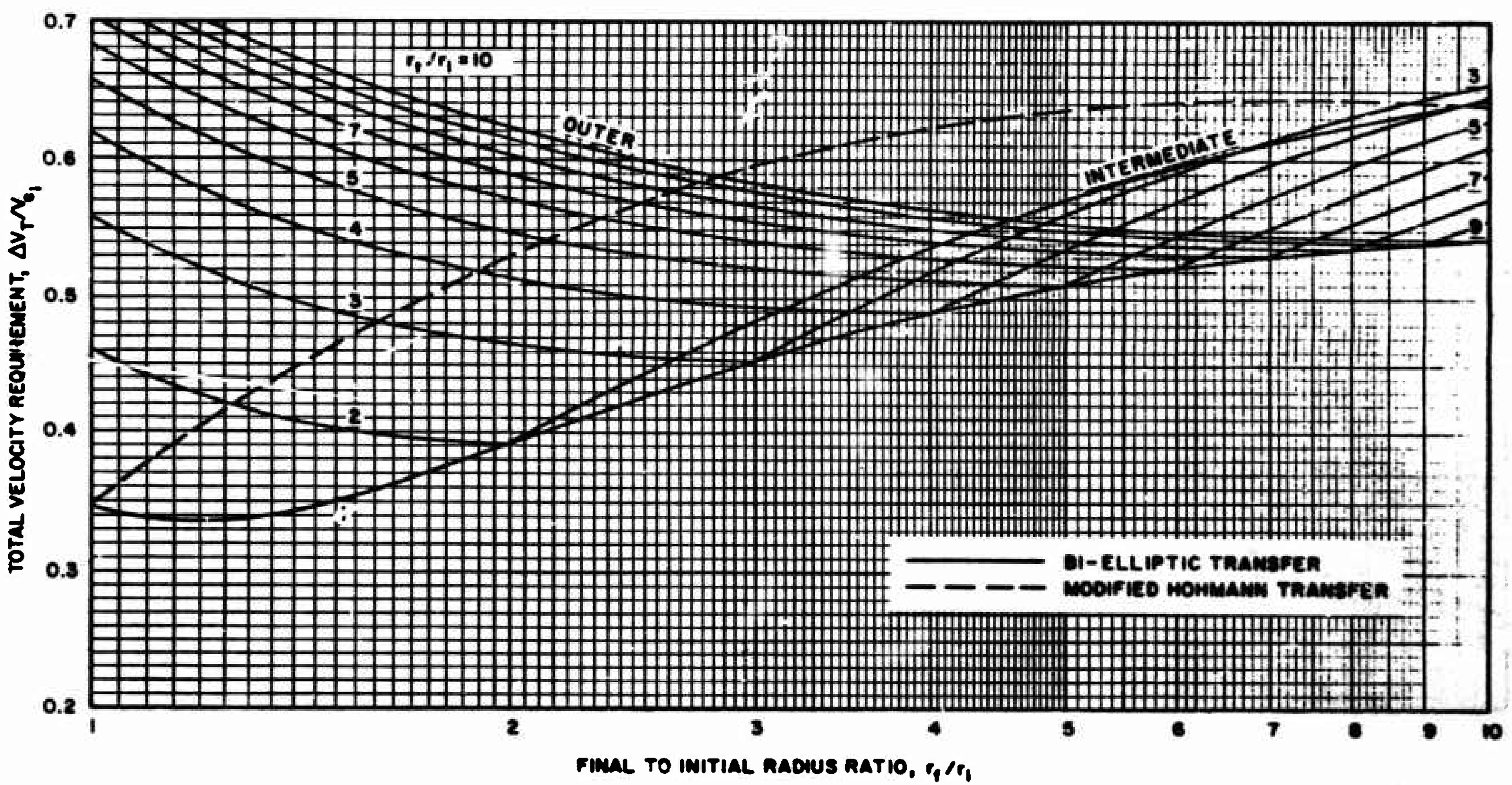

Figure 12. Minimum Bi-elliptic Velocity Requirements When θ = 20°

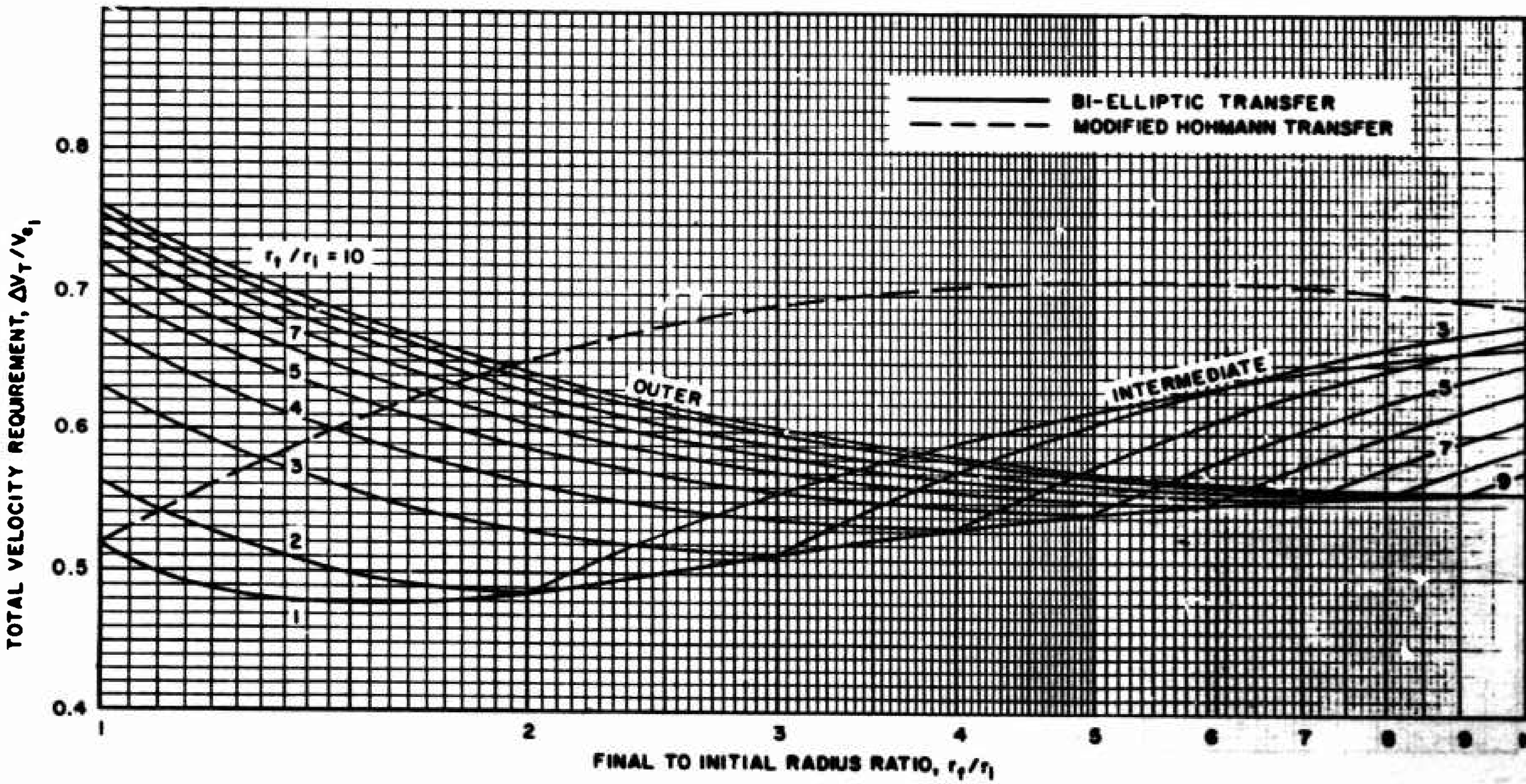

Figure 13. Minimum Bi-elliptic Velocity Requirements ...

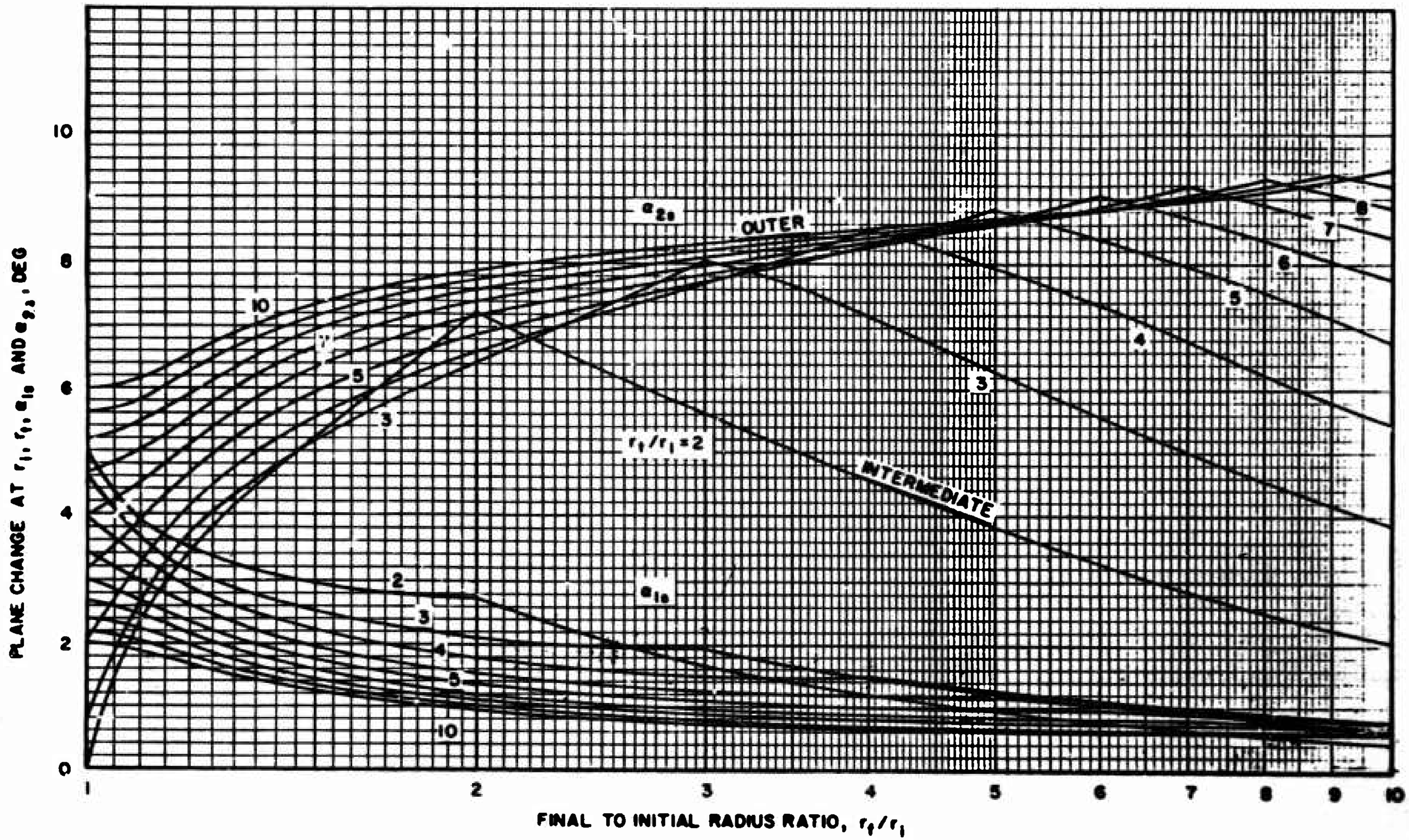

Figure 14. Division of Bi-elliptic Plane Change for Minimum Velocity When $\theta = 10^\circ = a_{1s} + a_{2s} + a_{3s}$

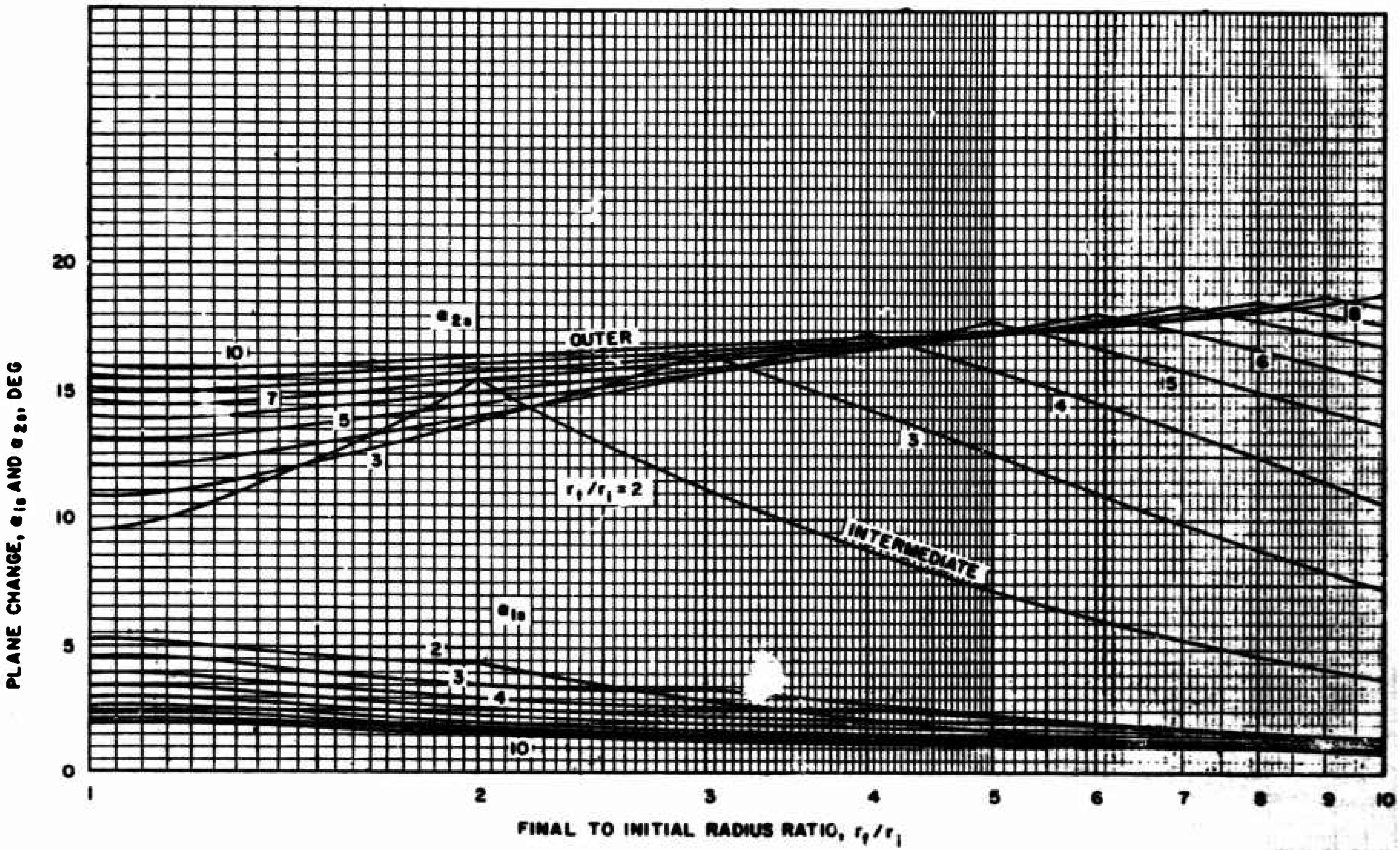

Figure 15. Division of Bi-elliptic Plane Change for Minimum Velocity When $\theta = 20° = a_{1s} + a_{2s} + a_{3s}$

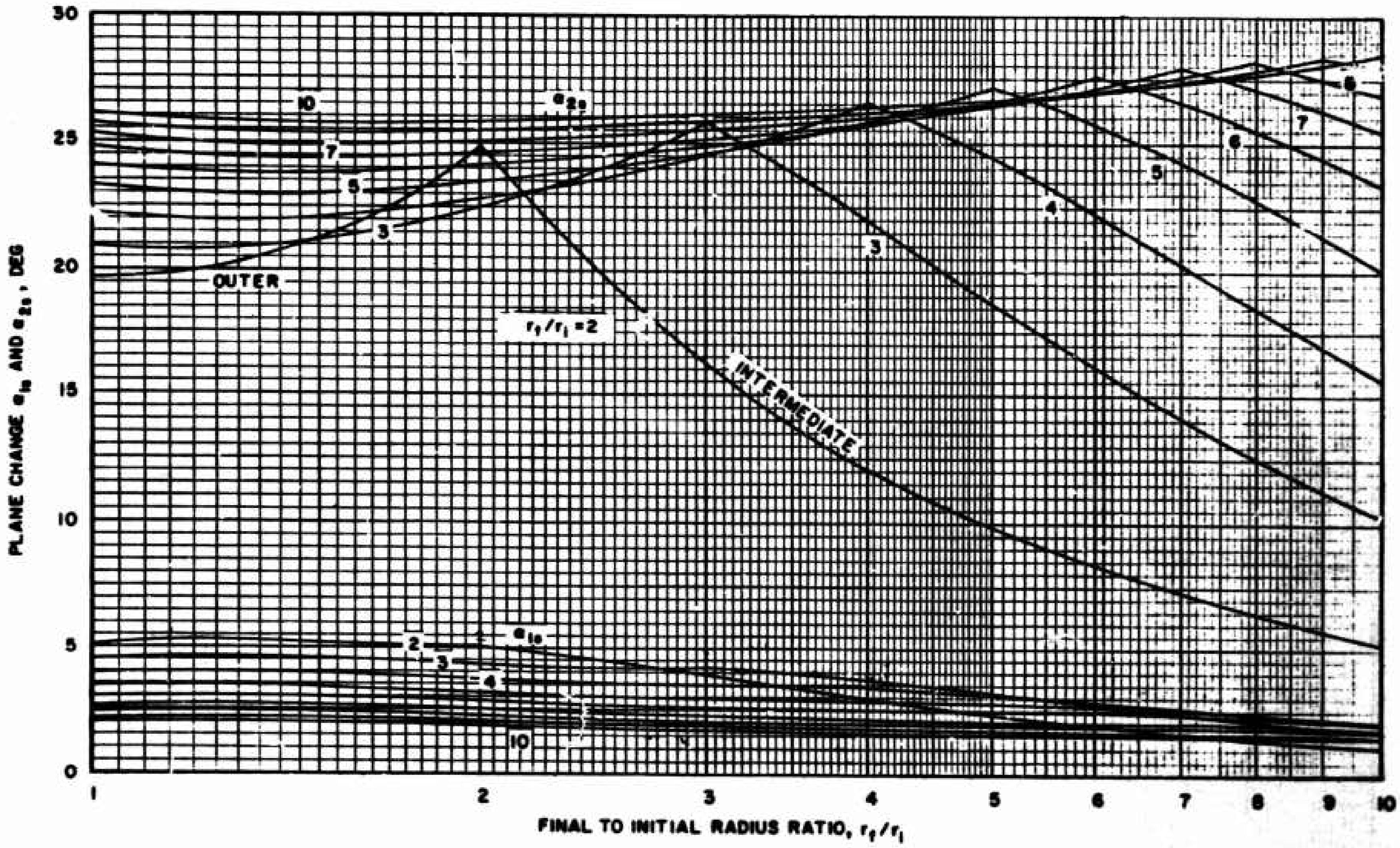

Figure 16. Division of Bi-elliptic Plane Change for Minimum Velocity When $\theta = 30° = a_{1s} + a_{2s} + a_{3s}$

mode is included in the figures for comparison purposes. When $\theta = 0°$, the modified Hohmann transfer is equivalent to the simple Hohmann transfer ($r_t/r_i = 1$).

Figures 14 through 16 show the values of $\alpha_1 = \alpha_{1s}$ and $\alpha_2 = \alpha_{2s}$ which yield the minimum values of ΔV_T when θ is respectively equal to 10°, 20°, and 30°. As in the previous graphs, only the outer and intermediate bi-elliptic transfers are considered.

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13 ABSTRACT

By the calculations presented, the minimum total velocity increment required for bi-elliptic transfer between non-coplanar circular orbits is obtained. The maneuver considered is the following: A vehicle in circular orbit at altitude h_i (radius r_i) applies an impulsive velocity ΔV_1 at the line of nodes. The effect of the application of ΔV_1 causes a plane change of amount α_1 and a transfer ellipse to a given transfer altitude h_t (radius r_t) is established. When the vehicle reaches h_t, a second impulsive velocity change ΔV_2 simultaneously changes the plane by amount α_2 and initiates a transfer ellipse to the altitude h_f (radius r_f) of the target orbit. A last impulse ΔV_3 changes the plane by amount α_3 and circularizes the orbit at altitude h_f, placing the vehicle in the final (target) circular orbit.

Studies were made of the choice of plane change angles α_1, α_2, and α_3, which minimizes $\Delta V_T \equiv \Delta V_1 + \Delta V_2 + \Delta V_3$ for given values of h_i, h_t, h_f and total plane change angle $\theta = \alpha_1 + \alpha_2 + \alpha_3$. Several limiting relations were obtained for α_1, α_2, and α_3; they are dependent on either the ratio r_t/r_i alone, or the ratios r_t/r_i and r_t/r_f, and are independent of θ.

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